1 ADT: Map

A *map* (also called *associative array*) is a collection of unique keys and a collection of values, where each key is associated with a single value. Supported operations are:

- creating an empty map;
- querying whether a map contains a given key;
- lookup of a value associated with a given key, if the key is present;
- inserting a key and a value to be associated with it, if the key is *not* already present;
- removing a key (together with the associated value), if the key is present.

Design an abstract data type MAP that corresponds to the specification given above.

TYPES

MAP [K, V]

FUNCTIONS

- $new\colon MAP[K,V]$
- $has(m,k)\colon MAP[K,V] \times K \to BOOLEAN$
- $item(m,k)\colon MAP[K,V] \times K \not\rightarrow V$
- $put(m,k,x)\colon MAP[K,V] \times K \times V \not\rightarrow MAP[K,V]$
- $remove(m,k)\colon MAP[K,V] \times K \not\rightarrow MAP[K,V]$

PRECONDITIONS

P1 $item(m,k)$ require $has(m,k)$

P2 $put(m,k,x)$ require $\neg has(m,k)$

P3 $remove(m,k)$ require $has(m,k)$

AXIOMS

A1 $\neg has(new,k)$

A2 $has(put(m,k,x),k)$

A3 $has(put(m,k,x),l) = has(m,l), \text{if } l \neq k$

A4 $\neg has(remove(m,k),k)$

A5 $has(remove(m,k),l) = has(m,l), \text{if } l \neq k$

A6 $item(put(m,k,x),k) = x$

A7 $item(put(m,k,x),l) = item(m,l), \text{if } l \neq k \land has(m,l)$

A8 $item(remove(m,k),l) = item(m,l), \text{if } l \neq k \land has(m,l)$

1.1 Proof of sufficient completeness

Prove that your ADT is sufficiently complete.

For all terms T of type MAP there exist resulting terms not involving any functions of the ADT when evaluating $has(T, k)$ and $item(T, k)$.

Induction basis

For all creators above holds.

- $has(new, k) \overset{A1}{=} False$

We don't have to check $item(new, k)$, because the precondition is never satisfied.

Induction hypothesis

Assume for any T_{sub} being a subterm of T that this is true.

Induction step

- For $T = put(T_{sub}, k, x)$:

$$has(put(T_{sub}, k, x), l) = \begin{cases} \overset{A2}{=} True \text{ if } k = l \\ \overset{A3}{=} has(T_{sub}, l) \text{ if } k \neq l \end{cases}$$

$$item(put(T_{sub}, k, x), l) = \begin{cases} \overset{A6}{=} x \text{ if } k = l \\ \overset{A7}{=} item(T_{sub}, l) \text{ if } k \neq l \wedge has(T_{sub}, l) \end{cases}$$

- For $T = remove(T_{sub}, k)$:

$$has(remove(T_{sub}, k), l) = \begin{cases} \overset{A4}{=} False \text{ if } k = l \\ \overset{A5}{=} has(T_{sub}, l) \text{ if } k \neq l \end{cases}$$

$$item(remove(T_{sub}, k), l) \overset{A8}{=} has(T_{sub}, l) \text{ if } k \neq l \wedge has(T_{sub}, l)$$