This is the proof as done with the help of the class, on-the-spot. It was spur-of-the-moment and hasn’t been checked in any deep way. For a systematic treatment see "Object-Oriented Software Construction, second edition", pages 156-159.

Example of a stack expression:

\[ e = \text{put} ( \text{put} ( \text{put} ( \text{remove} ( \text{put} ( \text{new}, x), y)...) ) \]

Property to prove:

For any well-formed stack expression \( e \), the axioms are powerful enough to yield the value of: \( \text{empty} (e) \)

Proof:

Induction on \( n \): number of parenthesis pairs in a well-formed stack expression.

1) Base step:

Prove induction hypothesis for \( n = 0 \).

\[ e = \text{new} \]
\[ \text{empty} (e) = \text{True} \]

PROVED!

2) Induction step:

Assume, for \( n > 0 \), that for any expression \( f \) of size \( < n \), we can compute:

\[ \text{empty} (f) \]

We must prove that for any expression \( e \) of size \( n \), we can compute:

\[ \text{empty} (e) \] if this is a well-formed expression.

Let \( e \) be an expression of size \( n > 0 \)
We are interested in \( \text{empty} (e) \)

\( e \) is of one of the forms

\[ \text{item} (f) \quad \text{-- empty} (e) \text{ is not well-formed} \]
\[ \text{empty} (f) \quad \text{-- empty} (e) \text{ is not well-formed} \]
\[ \text{put} (f, x) \]
\[ \text{remove} (f) \]

where \( f \) is an expression of size \( n - 1 \)

Case 1: \( e = \text{put} (f, x) \)
Then axiom 4 says \( \text{empty} (\text{put} (f, x)) = \text{false} \) OK!
Case 2: $e = \text{remove}(f)$

Then the number of parenthesis pairs of $e$ is $m + 1$

where $m$ = the number of parenthesis pairs of $f$

**LEMMA:**

In a correct expression (well-formed of course),

$\#\text{remove} \leq \#\text{put}$

END OF LEMMA

From the lemma, we know there is at least one "put" in $f$

Consider an outermost such "put"

It is enclosed in a "remove"

So $e$ has a subexpression of the form $\text{remove}(\text{put}(g, x))$

(which can be $e$ itself, or a proper subexpression)

OK, so we can replace that subexpression by $g$

without changing the value of $e$

Let $e'$ be the result of that change

We know that $e = e'$ from axiom 2

$e'$ has $m-1 = n-2$ parenthesis pairs

For which the result holds thanks to the induction hypothesis.

End.