ETH Zürich
Chair of Software Engineering

Software Verification

Lecture 13: Verification of Real-time Systems

Carlo A. Furia

Program Verification: the very idea

P: a program

```
max (a, b: INTEGER): INTEGER
    do
        if a > b then
            Result := a
        else
            Result := b
        end
    end
```

S: a specification

```
require
    true

ensure
    Result >= a
    Result >= b
```

Does $P \models S$ hold?

The Program Verification problem:

- Given: a program P and a specification S
- Determine: if every execution of P, for every value of input parameters, satisfies S

Real-time Verification

P: a program

```
max (a, b: INTEGER): INTEGER
    do
        if a > b then
            Result := a
        else
            Result := b
        end
    end
```

S: a specification

```
ensure
    Result >= a
    Result >= b

ensure -- real-time
"max terminates no sooner
 than 3 ms and no later than
 10 ms after invocation"
```

Does $P \vDash S$ hold?

The Real-time Verification problem:

- Given: program P (embedded in environment E) and real-time specification S
- Determine: if every execution of P (within E) satisfies S

Real-time Programs and Systems

Def. Real-time specification: specification that includes exact timing information.

Def. Real-time computation: computation whose specification is real-time. In other words: computation whose correctness depends not only on the value of the result but also on when the result is available.

- The timing of a piece of software is usually dependent on the environment where the computation takes place
- Hence, in real-time verification the focus shifts from programs to (software-intensive) systems
- The purely computational aspects can often be analyzed in isolation
- Real-time verification can then focus on real-time aspects of the system
  - e.g., synchronization, deadlines, delays, ...

while abstracting away most of the rest

Decidability vs. Expressiveness Trade-Off

The Real-time Verification problem:

- Given: program P (embedded in environment E) and real-time specification S
- Determine: if every execution of P (within E) satisfies S

P: a system ⇕ F(P): formal model of P

S: a real-time specification ⇕ N(S): formal annotation for S

Does $F(P) \models N(S)$ hold?

- The classes of F(P) and N(S) should guarantee:
  - enough expressiveness to include a quantitative notion of time
  - decidability of the verification problem

Real-time Model-Checking

The Real-time Model Checking problem:

- Given: a timed automaton A and a metric temporal-logic formula F
- Determine: if every run of A satisfies F or not
  - if not, also provide a counterexample: a run of A where F does not hold

A: a timed automaton $A \overset{?}{\models} F$ F: a metric temporal-logic formula

- The model-checking paradigm is naturally extended to real-time systems
- Different choices are possible for the family of automata and of formulae
  - The linear vs. branching time dichotomy is usually not significant for real-time
    - linear time is almost invariably preferred
  - A different attribute of time that becomes relevant in quantitative models is discrete vs. dense time

Discrete vs. dense (continuous) time

Discrete time

- sequence of isolated "steps"
- every instant has a unique successor
- e.g.: the naturals N = {0, 1, 2, ...}

+ simple and intuitive
+ verification usually decidable (and acceptably complex)
+ robust and elegant theoretical framework

– cannot model true asynchrony
– unsuitable to model physical variables

Dense (or continuous) time

- arbitrarily small distances
- the successor of an instant is not defined
- e.g.: the reals R

+ can model true asynchrony
+ accurate modeling of physical variables

– tricky to understand
– verification often undecidable (or highly complex)
– lacks a unifying framework

Discrete Real-time Model-Checking

Timed Automata and Metric Temporal Logic

Discrete Real-time Model-Checking

Discrete real-time model checking extends standard "untimed" model checking straightforwardly:

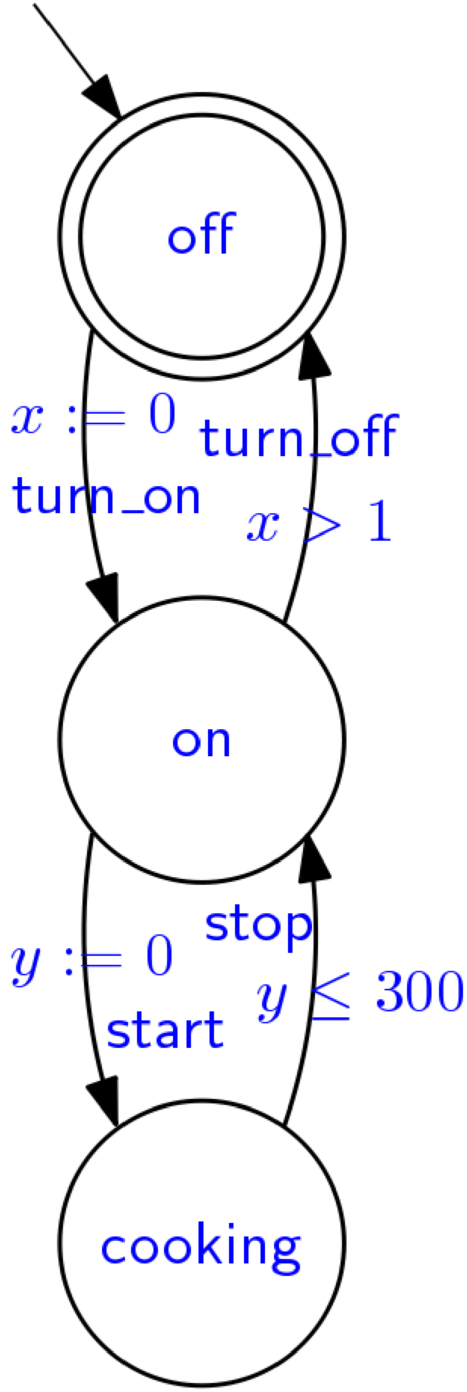
- Discrete Timed Automata (TA) extend the Finite-State Automata (FSA)
- Metric Temporal Logic (MTL) extends Linear Temporal Logic (LTL)

The Discrete Real-time Model Checking problem:

- Given: a discrete TA A and an MTL formula F
- Determine: if every run of A satisfies F or not
  - if not, also provide a counterexample: a run of A where F does not hold

A: a discrete TA $\quad A \models^{?} F \quad$ F: an MTL formula

Timed Automata: Syntax



Timed Automata: Syntax

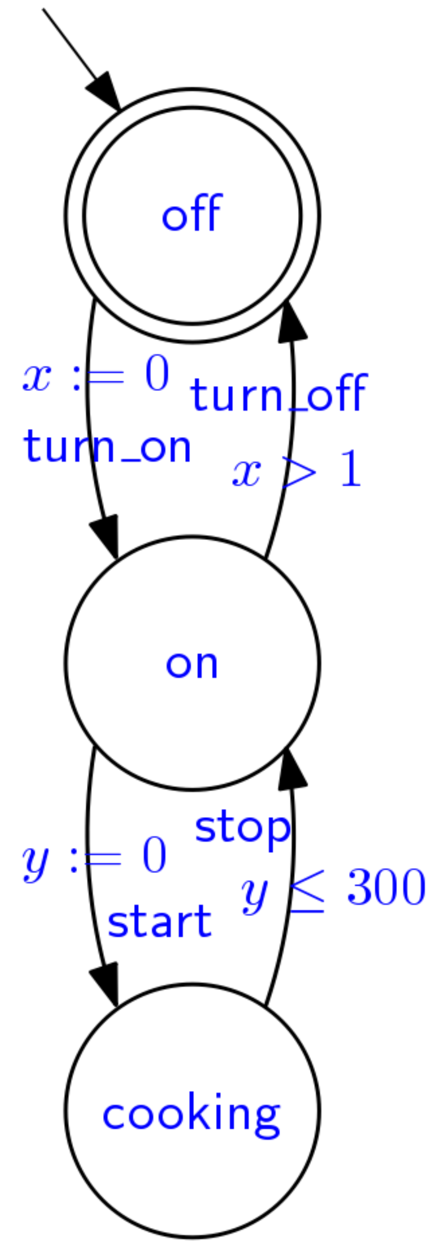
Def. Nondeterministic Timed Automaton (TA)

A tuple [Σ, S, C, I, E, F]:

- Σ: finite nonempty (input) alphabet
- S: finite nonempty set of locations (i.e., discrete states)
- C: finite set of clocks
- I, F: set of initial/final states
- E: finite set of edges [s, σ, c, ρ, s']
  - s ∈ S: source location
  - s' ∈ S: target location
  - σ ∈ Σ: input character (also "label")
  - c: clock constraint in the form:

    c ::= x ≈ k | ¬ c | c1 ∧ c2

    - x, y ∈ C are clocks
    - k ∈ N is an integer constant
    - ≈ is a comparison operator among <, ≤, >, ≥, =
  - ρ ⊆ C: set of clock that are reset (to 0)



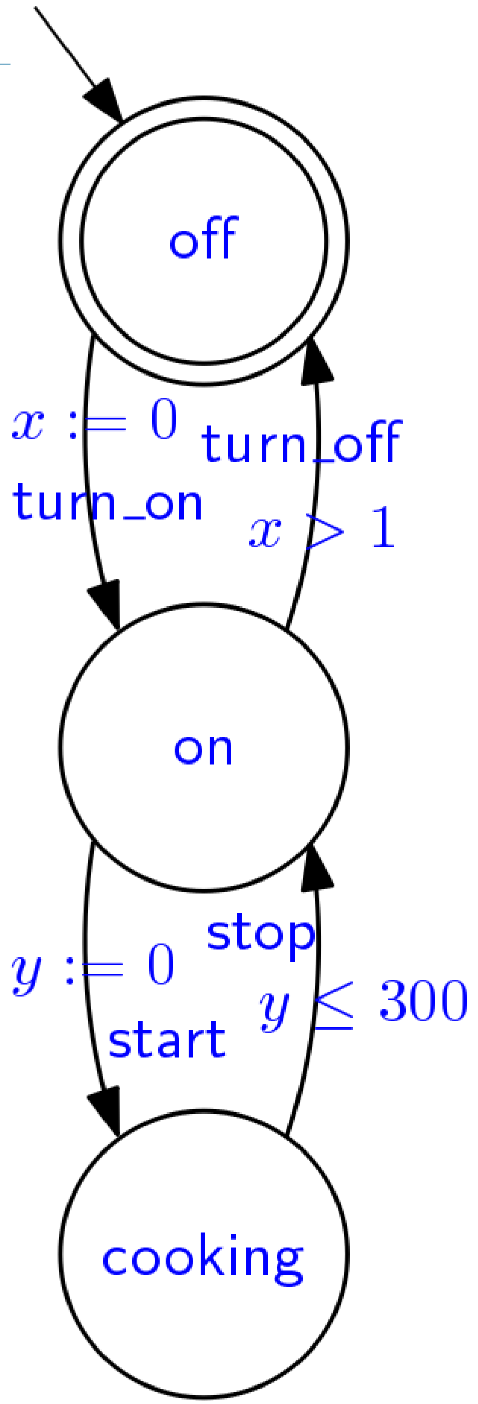
Timed Automata: Semantics

Accepting run:

r = [off, (x=0, y=0)]
[on, (x=0, y=3)]
[cooking, (x=8, y=0)]
[on, (x=81, y=73)]
[off, (x=85, y=77)]

Over input timed word:

w = [turn_on, 3]
[start, 11]
[stop, 84]
[turn_off, 88]



Timed Automata: Semantics

Def. A timed word $w = w(1)\ w(2) \ldots w(n) \in (\Sigma \times N)^*$ is a sequence of pairs $[\sigma(i), t(i)]$ such that:

- the sequence of timestamps $t(1), t(2), \ldots, t(n)$ is increasing
- $[\sigma(i), t(i)]$ represents the i-th character $\sigma(i)$ read at time $t(i)$

Def. An accepting run of a TA $A=[\Sigma, S, C, I, E, F]$ over input timed word $w = [\sigma(1), t(1)] \ldots [\sigma(n), t(n)] \in (\Sigma \times N)^*$ is a sequence $r = [s(0), v(0,1), \ldots, v(0,|C|)] \ldots [s(n), v(n,1), \ldots, v(n,|C|)] \in (S \times N^{|C|})^*$ of (extended) states such that:

- it starts from an initial and ends in an accepting state: $s(0) \in I$, $s(n) \in F$
- initially all clocks are reset to 0: $v(0,k) = 0$ for all $1 \leq k \leq |C|$
- for every transition $(0 \leq i < n)$:
  $[\ s(i)\ v(i,1) \ldots v(i,|C|)\ ] \dashrightarrow [\ s(i+1)\ v(i+1,1) \ldots v(i+1,|C|)\ ]$
  some edge $[s(i), \sigma(i+1), c, \rho, s(i+1)]$ in E is followed:
  - the clock values $v(i,1) + (t(i+1) - t(i)) \ldots v(i,|C|) + (t(i+1) - t(i))$ satisfy the constraint c
  - $v(i+1,k) =$ if k-th clock is in ρ then 0 else $v(i,k) + t(i+1) - t(i)$

Timed Automata: Semantics

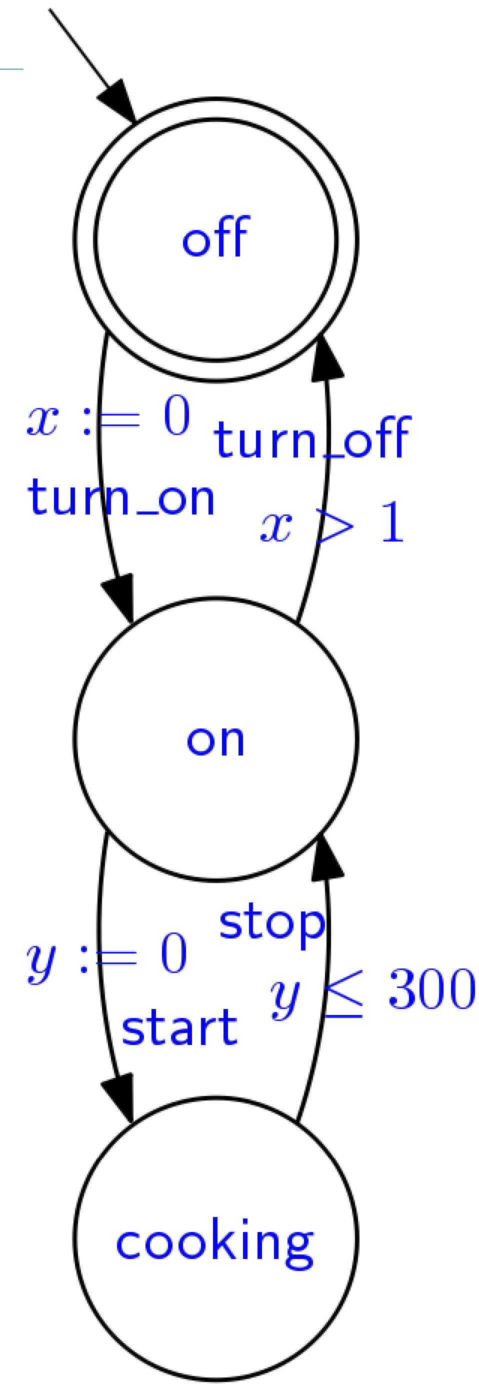
Def. Any TA A=[Σ, S, C, I, E, F] defines
a set of input timed words ⟨A⟩:
⟨A⟩ ≜ { w ∈ (Σ x N)* | there is
an accepting run of A
over w }

⟨A⟩ is called the language of A

With regular expressions and arithmetic:

⟨A⟩ = ([turn_on, t_1]
([start, t_2] [stop, t_3])*
[turn_off, t_4])*

with $t_3 - t_2 \leq 300$ and $t_4 - t_1 > 1$



Metric (Linear) Temporal Logic

<>[2,4) stop

"there is an occurrence of stop between 2 (included) and 4 (excluded) time units in the future"

- [any, t ≤ 1]* [stop, 2] [stop, 3] [any, 4] [any, 7] ...
- [any, t < 3]* [stop, 3] [any, 4] [any, t > 4] ...

[](2,4] start

"start holds between 2 (excluded) and 4 (included) time units in the future"

- [any, 0] [any, 1] [any, 2] [start, 3] [start, 4] [any, t > 4]*
- [any, 0] [any, 1] [any, 2] [start, 3] [any, t > 4]*
- [stop, 0] [stop, 1]

Metric (Linear) Temporal Logic

[] (start ⇒ <>(3,10] stop)

"every occurrence of start is followed by an occurrence of stop between 3 (excluded) and 10 (included) time units in the future"

cook U(3,10] stop

"stop occurs between 3 (excluded) and 10 (included) time units in the future, and cook holds until then"

Metric (Linear) Temporal Logic: Syntax

Def. Propositional Metric Temporal Logic (MTL) formulae:

$$F \;::=\; p \;\mid\; \neg F \;\mid\; F \wedge G \;\mid\; F\; U\langle a,b\rangle\; G$$

with $p \in P$ any atomic proposition and $\langle a,b\rangle$ an interval of the time domain (w.l.o.g. with integer endpoints).

Temporal (modal) operators:

- next: $X\,F \triangleq True\; U[1,1]\; F$
- bounded until: $F\; U\langle a,b\rangle\; G$
- bounded release: $F\; R\langle a,b\rangle\; G \triangleq \neg\,(\neg F\; U\langle a,b\rangle\; \neg G)$
- bounded eventually: $\langle\rangle\langle a,b\rangle\; F \triangleq True\; U\langle a,b\rangle\; F$
- bounded always: $[\,]\langle a,b\rangle\; F \triangleq \neg\, \langle\rangle\langle a,b\rangle\; \neg F$
- intervals can be unbounded; e.g., $[3, \infty)$
- intervals with pseudo-arithmetic expressions; e.g.:
  - ≥ 3 for $[3, \infty)$
  - $= 1$ for $[1,1]$
  - $[0, \infty)$ is simply omitted

Metric Temporal Logic: Semantics

Def. A timed word $w = [\sigma(1), t(1)]\, [\sigma(2), t(2)] \dots [\sigma(n), t(n)] \in (P \times N)^*$ satisfies LTL formula F at position $1 \le i \le n$, denoted $w, i \vDash F$, when:

- $w, i \vDash p$ iff $p = \sigma(i)$
- $w, i \vDash \neg F$ iff $w, i \vDash F$ does not hold
- $w, i \vDash F \wedge G$ iff both $w, i \vDash F$ and $w, i \vDash G$ hold
- $w, i \vDash F\, U\langle a,b\rangle\, G$ iff for some $i \le j \le n$ such that $t(j) - t(i) \in \langle a,b\rangle$ it is: $w, j \vDash G$ and for all $i \le k < j$ it is $w, k \vDash F$
  - i.e., F holds until G will hold within <a, b>

For derived operators:

- $w, i \vDash \langle\rangle\langle a,b\rangle\, F$ iff for some $i \le j \le n$ such that $t(j) - t(i) \in \langle a,b\rangle$ it is: $w, j \vDash F$
  - i.e., F holds eventually within <a,b>
- $w, i \vDash [\,]\langle a,b\rangle\, F$ iff for all $i \le j \le n$ such that $t(j) - t(i) \in \langle a,b\rangle$ it is: $w, j \vDash F$
  - i.e., F holds always within <a,b>

Metric Temporal Logic: Semantics

Def. Satisfaction:

$$w \models F \quad \triangleq \quad w, 1 \models F$$

i.e., timed word w satisfies formula F initially

Def. Any MTL formula F defines a set of timed words ⟨F⟩:

$$\langle F \rangle \triangleq \{ w \in (P \times N)^* \mid w \models F \}$$

⟨F⟩ is called the language of F

Discrete Real-time Model-Checking

From Real-time to Untimed Model-Checking

Discrete-time Real-time Model Checking

An semantic view of the Real-time Model Checking problem:

Given: a timed automaton A and an MTL formula F

- if $\langle A \rangle \cap \langle \neg F \rangle$ is empty then every run of A satisfies F
- if $\langle A \rangle \cap \langle \neg F \rangle$ is not empty then some run of A does not satisfy F
  - any member of the nonempty intersection $\langle A \rangle \cap \langle \neg F \rangle$ is a counterexample

How to check $\langle A \rangle \cap \langle \neg F \rangle = \emptyset$ algorithmically (given A, F)?

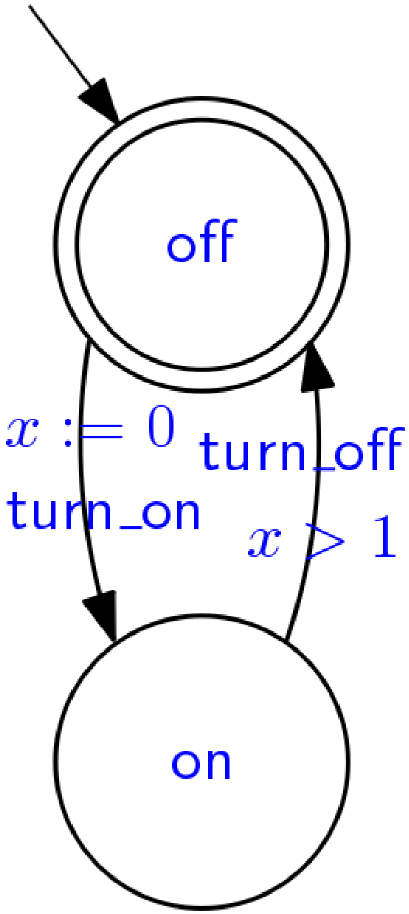
For a discrete time domain we can reduce real-time model checking to (untimed) model-checking:

- Transform timed automaton A into finite-state automaton A'
- Transform MTL formula F into LTL formula F'

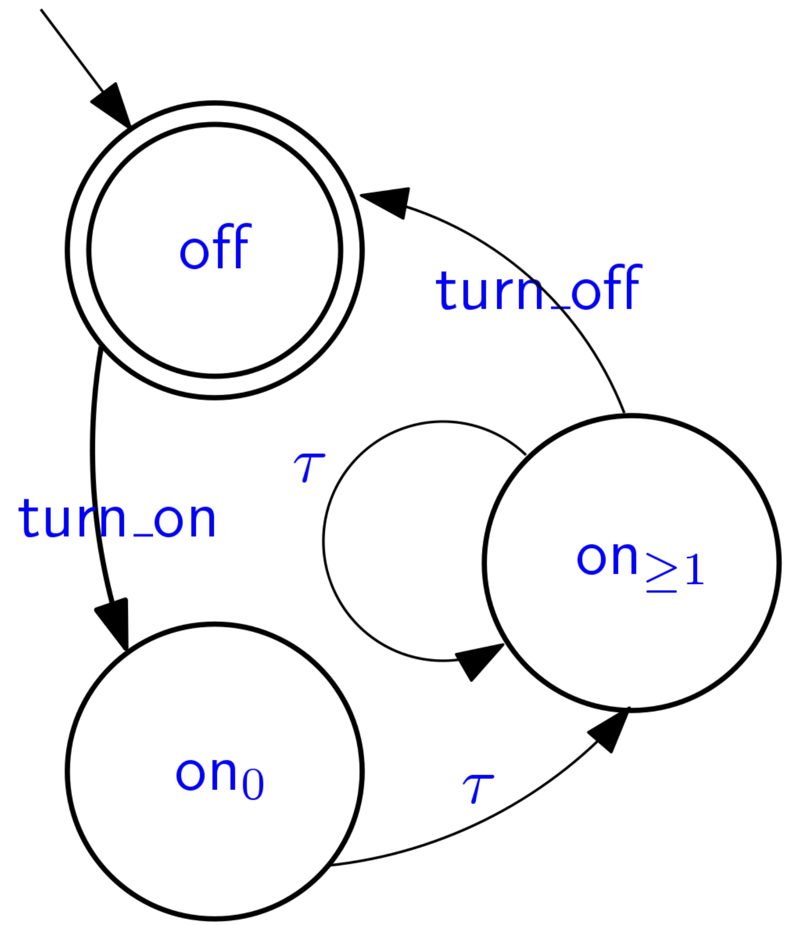
  $\langle A \rangle \cap \langle \neg F \rangle = \emptyset$ iff $\langle A' \rangle \cap \langle \neg F' \rangle = \emptyset$
- Re-use standard model-checking algorithms

Reduce discrete-time TAs to FSAs

Use states of an FSA to "count" discrete time steps according to the semantics of the TA



- transitions with special events τ are time steps without events.
- on_0 represents location on with clock x = 0
- $\text{on}_{\geq 1}$ represents location on with clock x ≥ 1



Reduce discrete-time MTL to LTL

Use next operator X to "count" discrete time steps according to the semantics of the MTL formula

- <>[1,3] p becomes Xp ∨ XXp ∨ XXXp
  - More compactly X(p ∨ X(p ∨ Xp))
- []≥5 p becomes X^5 [](p ∨ ⊤)
  - X^5p is a shorthand for XXXXXp
  - The disjunction is needed because we may have time increments without events
- The encoding for bounded until is a bit more intricate but not different in principle

Discrete-time Real-time MC: Complexity

There is an exponential blow-up in complexity when switching from (untimed) linear-time model checking to discrete-time real-time model checking:

- Discrete-time real-time MTL model checking: EXPSPACE-complete
  - in practice: double-exponential time
- LTL model checking: PSPACE-complete
  - in practice: singly-exponential time
- The blow up occurs only if the constants (in timed automata and MTL formulas) are encoded succinctly in binary
  - blow-up due to the "unrolling" of binary constants as FSA states or nested next operators

Dense Real-time Model-Checking

Timed Automata and Metric Temporal Logic

Dense Real-time Model-Checking

Dense real-time model checking considers the same model as discrete real-time model checking but with $\mathbb{R}_{\geq 0}$ as time domain:

- A dense Timed Automaton (TA) models the system
- Dense-time Metric Temporal Logic (MTL) models the property

- The syntax of TA and MTL need not be changed for dense time
  - with the possible exception of allowing fractional time bounds
- The semantics of TA and MTL is also unchanged except that:
  - $\mathbb{R}_{\geq 0}$ replaces $\mathbb{N}$ as time domain
  - Infinite words are considered by default:
    - This is a technicality that we will ignore in the presentation for simplicity, although it does affect some results. (See later for some details.)

Dense Real-time Model-Checking

Dense real-time model checking extends standard "untimed" model checking:

- Timed Automata (TA) extend Finite-State Automata (FSA)
- Metric Temporal Logic (MTL) extends Linear Temporal Logic (LTL)

The Dense Real-time Model Checking problem:

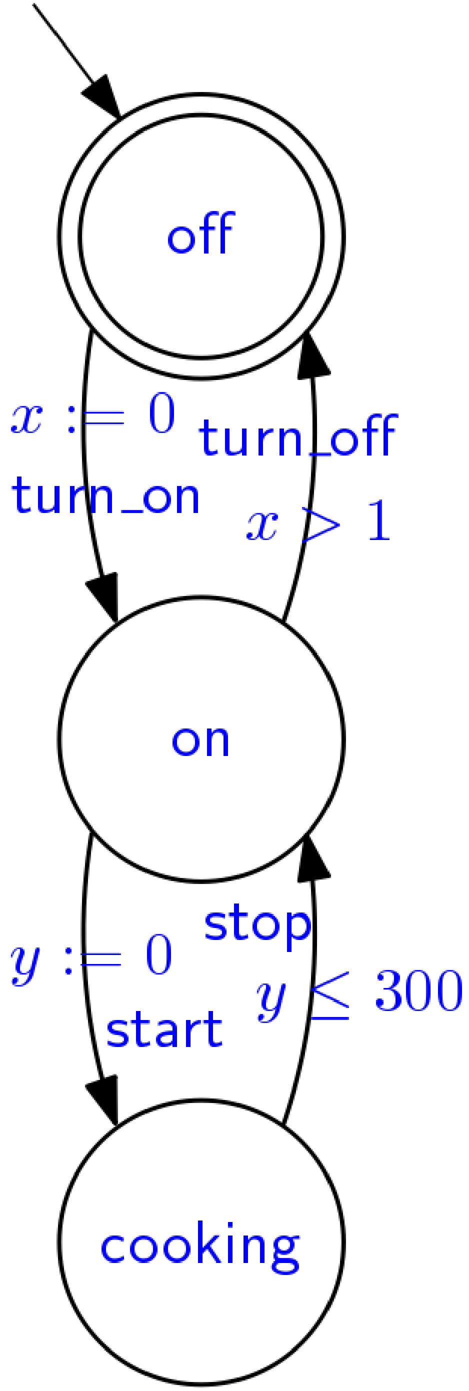
- Given: a dense TA A and an MTL formula F
- Determine: if every run of A satisfies F or not
  - if not, provide a counterexample: a run of A where F does not hold

A: a TA

$$A \overset{?}{\models} F$$

F: an MTL formula

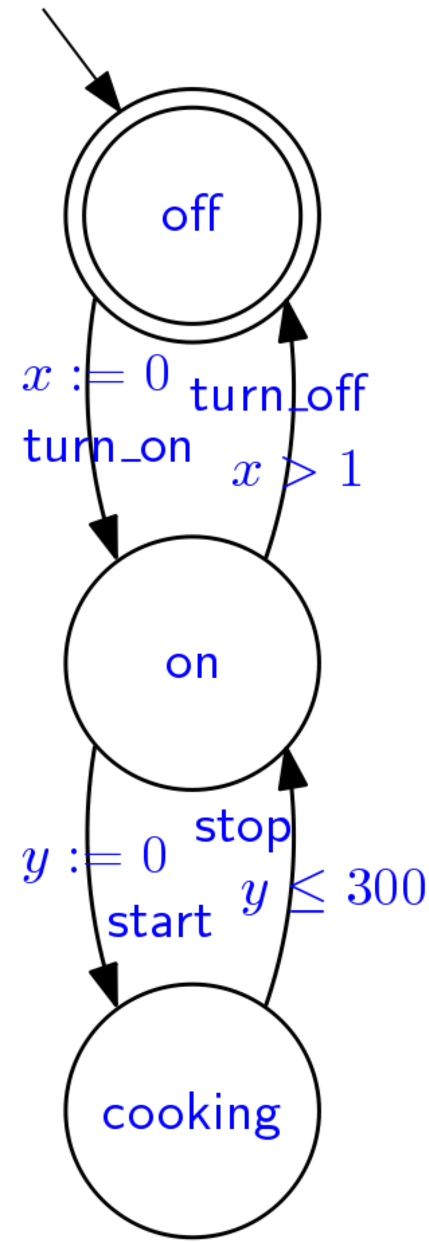
Timed Automata: Syntax



Timed Automata: Syntax

Def. Nondeterministic Timed Automaton (TA):
a tuple [Σ, S, C, I, E, F]:

- Σ: finite nonempty (input) alphabet
- S: finite nonempty set of locations (i.e., discrete states)
- C: finite set of clocks
- I, F: set of initial/final states
- E: finite set of edges [s, σ, c, ρ, s']
  - s ∈ S: source location
  - s' ∈ S: target location
  - σ ∈ Σ: input character (also "label")
  - c: clock constraint in the form:
    c ::= x ≈ k | ¬ c | c1 ∧ c2
    - x, y ∈ C are clocks
    - k ∈ N is an integer constant
    - ≈ is a comparison operator among <, ≤, >, ≥, =
  - ρ ⊆ C: set of clock that are reset (to 0)



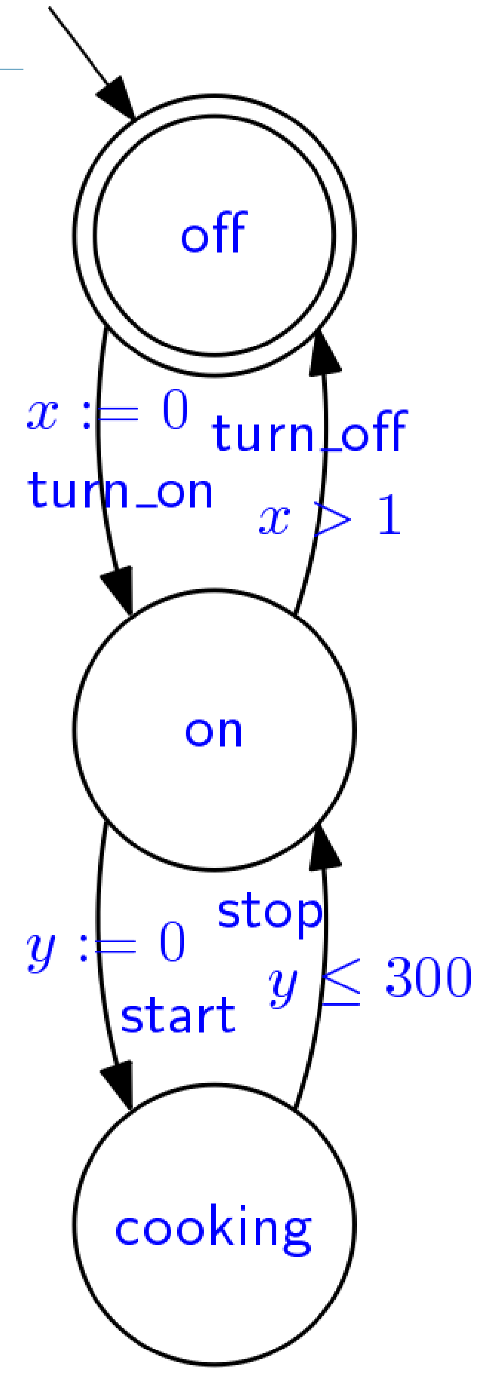
Timed Automata: Semantics

Accepting run:

r = [off, (x=0, y=0)]
[on, (x=0, y=3.2)]
[cooking, (x=8.5, y=0)]
[on, (x=81.7, y=73.2)]
[off, (x=84.91, y=76.41)]

Over input timed word:

w = [turn_on, 3.2]
[start, 11.7]
[stop, 84.9]
[turn_off, 88.11]



Timed Automata: Semantics

Def. A timed word $w = w(1)\, w(2) \ldots w(n) \in (\Sigma \times R)^*$ is a sequence of pairs $[\sigma(i), t(i)]$ such that:

- the sequence of timestamps $t(1), t(2), \ldots, t(n)$ is increasing
- $[\sigma(i), t(i)]$ represents the i-th character $\sigma(i)$ read at time $t(i)$

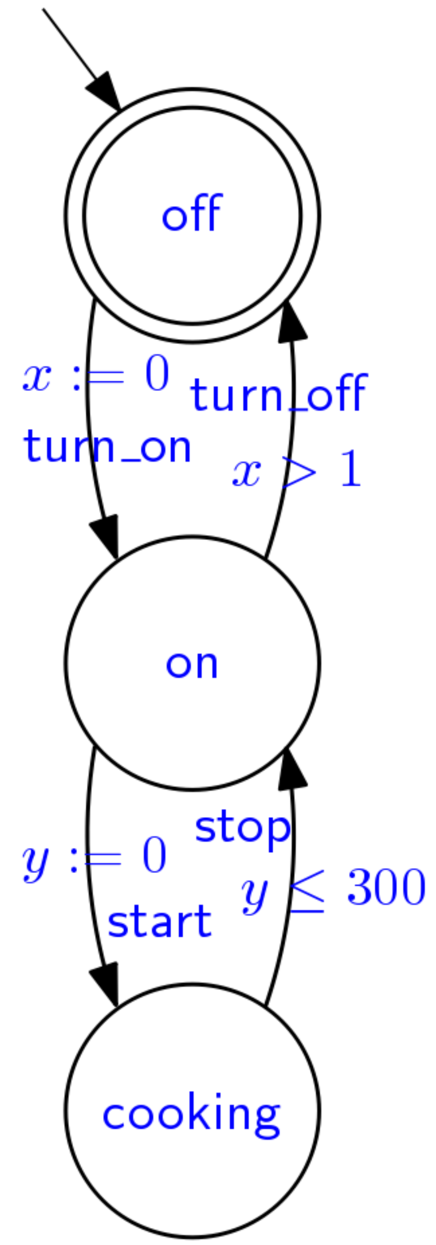
Def. An accepting run of a TA $A=[\Sigma, S, C, I, E, F]$ over input timed word
$w = [\sigma(1), t(1)] \ldots [\sigma(n), t(n)] \in (\Sigma \times R)^*$ is a sequence
$r = [s(0), v(0,1), \ldots, v(0,|C|)] \ldots [s(n), v(n,1), \ldots, v(n,|C|)] \in (S \times R^{|C|})^*$
of (extended) states such that:

- it starts from an initial and ends in an accepting state: $s(0) \in I$, $s(n) \in F$
- initially all clocks are reset to 0: $v(0,k) = 0$ for all $1 \leq k \leq |C|$
- for every transition $(0 \leq i < n)$:
  $[\ s(i)\ v(i,1) \ldots v(i,|C|)\ ] \dashrightarrow [\ s(i+1)\ v(i+1,1) \ldots v(i+1,|C|)\ ]$
  some edge $[s(i), \sigma(i+1), c, \rho, s(i+1)]$ in E is followed:
  - the clock values $v(i,1) + (t(i+1) - t(i)) \ldots v(i,|C|) + (t(i+1) - t(i))$ satisfy the constraint c
  - $v(i+1,k) =$ if k-th clock is in ρ then 0 else $v(i,k) + t(i+1) - t(i)$

Timed Automata: Semantics

Def. Any TA A=[Σ, S, C, I, E, F] defines
a set of input timed words ⟨A⟩:
⟨A⟩ ≜ { w ∈ (Σ x R)* | there is an accepting run of A over w }

⟨A⟩ is called the language of A



With regular expressions and arithmetic:

⟨A⟩ = ([turn_on, t_1] ([start, t_2] [stop, t_3])* [turn_off, t_4])*

with $t_3 - t_2 \le 300$ and $t_4 - t_1 > 1$

Metric (Linear) Temporal Logic

<>[2,4) stop

"there is an occurrence of stop between 2 (included) and 4 (excluded) time units in the future"

- [any, t < 2]* [stop, 2] [stop, 3] [any, 3.5] [any, 3.7] ...
- [any, t < 3.99]* [stop, 3.99] [any, 4] [any, t > 4] ...

[](2,4] start

"start holds between 2 (excluded) and 4 (included) time units in the future"

- [any, t ≤ 2] [start, 2.2] [start, 3] [start, 4] [any, t > 4] ...
- [any, t ≤ 2] [start, 4] [any, t > 4] ...
- [stop, 0] [stop, 0.3] [stop, 0.7]

Metric (Linear) Temporal Logic

[] (start ⇒ <>(3,10] stop)

"every occurrence of start is followed by an occurrence of stop between 3 (excluded) and 10 (included) time units in the future"

cook U(3,10] stop

"stop occurs between 3 (excluded) and 10 (included) time units in the future, and cook holds until then"

Metric (Linear) Temporal Logic: Syntax

Def. Propositional Metric Temporal Logic (MTL) formulae:

$$F ::= p \mid \neg F \mid F \wedge G \mid F\ U\langle a,b\rangle\ G$$

with $p \in P$ any atomic proposition and $\langle a,b\rangle$ an interval of the time domain (w.l.o.g. with integer endpoints).

Temporal (modal) operators:

- next: $X\ F \triangleq \text{True}\ U[1,1]\ F$
- bounded until: $F\ U\langle a,b\rangle\ G$
- bounded release: $F\ R\langle a,b\rangle\ G \triangleq \neg\ (\neg F\ U\langle a,b\rangle\ \neg G)$
- bounded eventually: $\langle\rangle\langle a,b\rangle\ F \triangleq \text{True}\ U\langle a,b\rangle\ F$
- bounded always: $[]\langle a,b\rangle\ F \triangleq \neg\ \langle\rangle\langle a,b\rangle\ \neg F$
- intervals can be unbounded; e.g., $[3, \infty)$
- intervals with pseudo-arithmetic expressions; e.g.:
  - ≥ 3 for $[3, \infty)$
  - $= 1$ for $[1,1]$
  - $[0, \infty)$ is simply omitted

Metric Temporal Logic: Semantics

Def. A timed word $w = [\sigma(1), t(1)]\,[\sigma(2), t(2)] \ldots [\sigma(n), t(n)] \in (P \times R)^*$ satisfies LTL formula F at position $1 \leq i \leq n$, denoted $w, i \models F$, when:

- $w, i \models p$ iff $p = \sigma(i)$
- $w, i \models \neg F$ iff $w, i \models F$ does not hold
- $w, i \models F \wedge G$ iff both $w, i \models F$ and $w, i \models G$ hold
- $w, i \models F\ U\langle a,b\rangle\ G$ iff for some $i \leq j \leq n$ such that $t(j) - t(i) \in \langle a,b\rangle$ it is: $w, j \models G$ and for all $i \leq k < j$ it is $w, k \models F$
  - i.e., F holds until G will hold within $\langle a, b\rangle$

For derived operators:

- $w, i \models \langle\rangle\langle a,b\rangle\ F$ iff for some $i \leq j \leq n$ such that $t(j) - t(i) \in \langle a,b\rangle$ it is: $w, j \models F$
  - i.e., F holds eventually within $\langle a,b\rangle$
- $w, i \models [\,]\langle a,b\rangle\ F$ iff for all $i \leq j \leq n$ such that $t(j) - t(i) \in \langle a,b\rangle$ it is: $w, j \models F$
  - i.e., F holds always within $\langle a,b\rangle$

Metric Temporal Logic: Semantics

Def. Satisfaction:

$$w \models F \quad \triangleq \quad w, 1 \models F$$

i.e., timed word w satisfies formula F initially

Def. Any MTL formula F defines a set of timed words ⟨F⟩:

$$\langle F \rangle \triangleq \{\, w \in (P \times R)^* \mid w \models F \,\}$$

⟨F⟩ is called the language of F

Dense Real-time Model-Checking

What's Decidable?

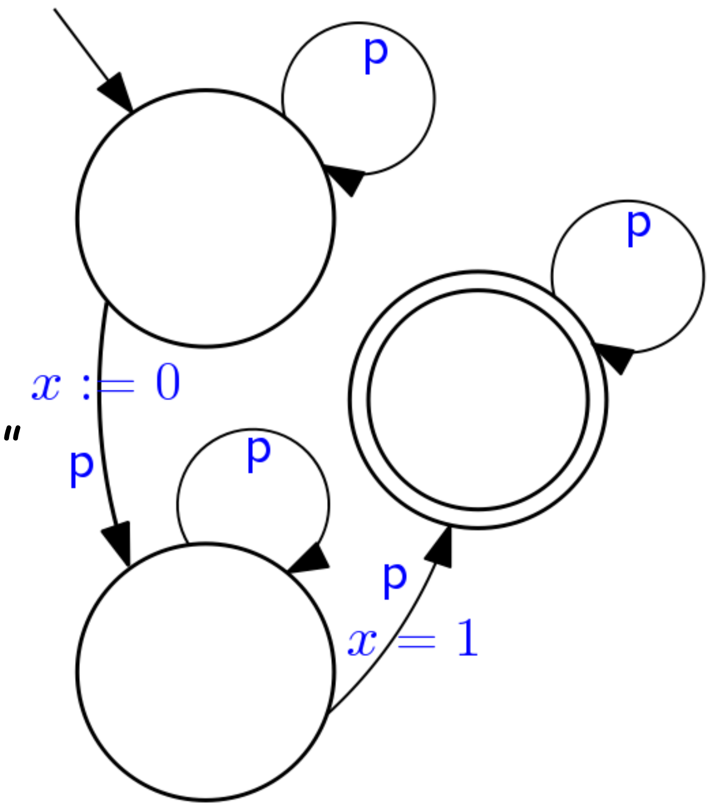
TAs not Closed under Complement

A: a dense TA $\quad A \stackrel{?}{\models} F \quad$ F: a dense-time MTL formula

Fundamental problem:

Dense timed automata are not closed under complement

The complement of the language of this TA isn't accepted by any TA:

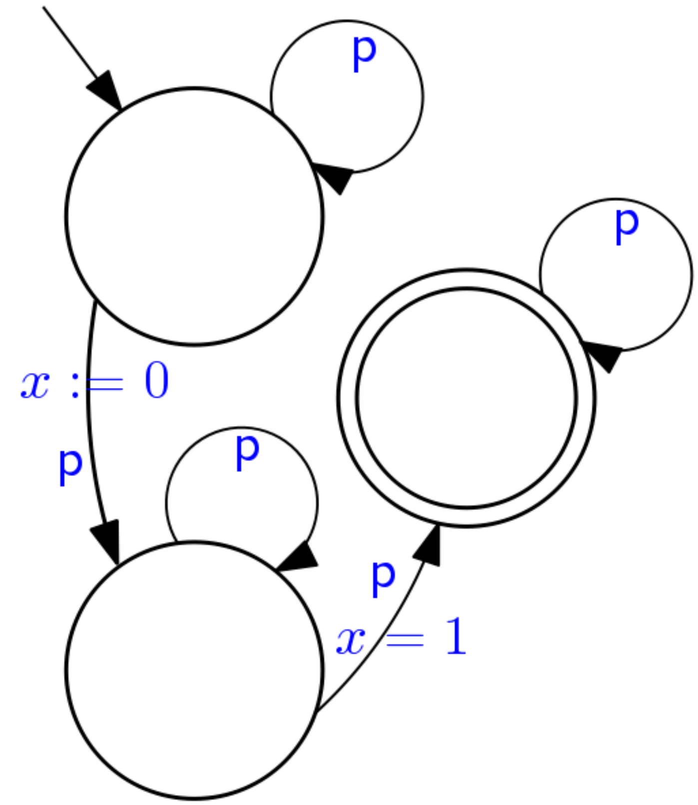


- language of this TA:
  "there exist two p's separated by one t.u."
- complement language:
  "no two p's are separated by one t.u."
- intuition: need a clock for each p within the past time unit, but there can be an unbounded number of such p's because time is dense

TAs not Closed under Complement

Fundamental problem:

- Dense TAs are not closed under complement
- MTL is clearly closed under complement
  - Language of the TA: <> (p ∧ <>=1 p)
  - Complement language of the TA: ¬ <> (p ∧ <>=1 p) = [] (p ⇒ ¬ <>=1 p)
- Hence, automata-theoretic dense real-time model-checking is unfeasible (in general)



Dense MTL Model Checking is Undecidable

An even more fundamental problem:

The dense-time model-checking problem for MTL and TAs is undecidable (for infinite words)

- no approach is going to work, not just the automata-theoretic one

MTL and TAs are "too expressive" over dense time

What's Decidable about Timed Automata

Let's revisit the three algorithmic components of automata-theoretic model checking:

- MTL2TA: given MTL formula F build TA a(F) such that ⟨F⟩ = ⟨a(F)⟩
  - undecidable problem (for infinite words)
- TA-Intersection: given TAs A, B build TA C such that ⟨A⟩ ∩ ⟨B⟩ = ⟨C⟩
  - decidable
- TA-Emptiness: given TA A check whether ⟨A⟩ = ∅ is the case
  - decidable!

Dense Real-time Model-Checking

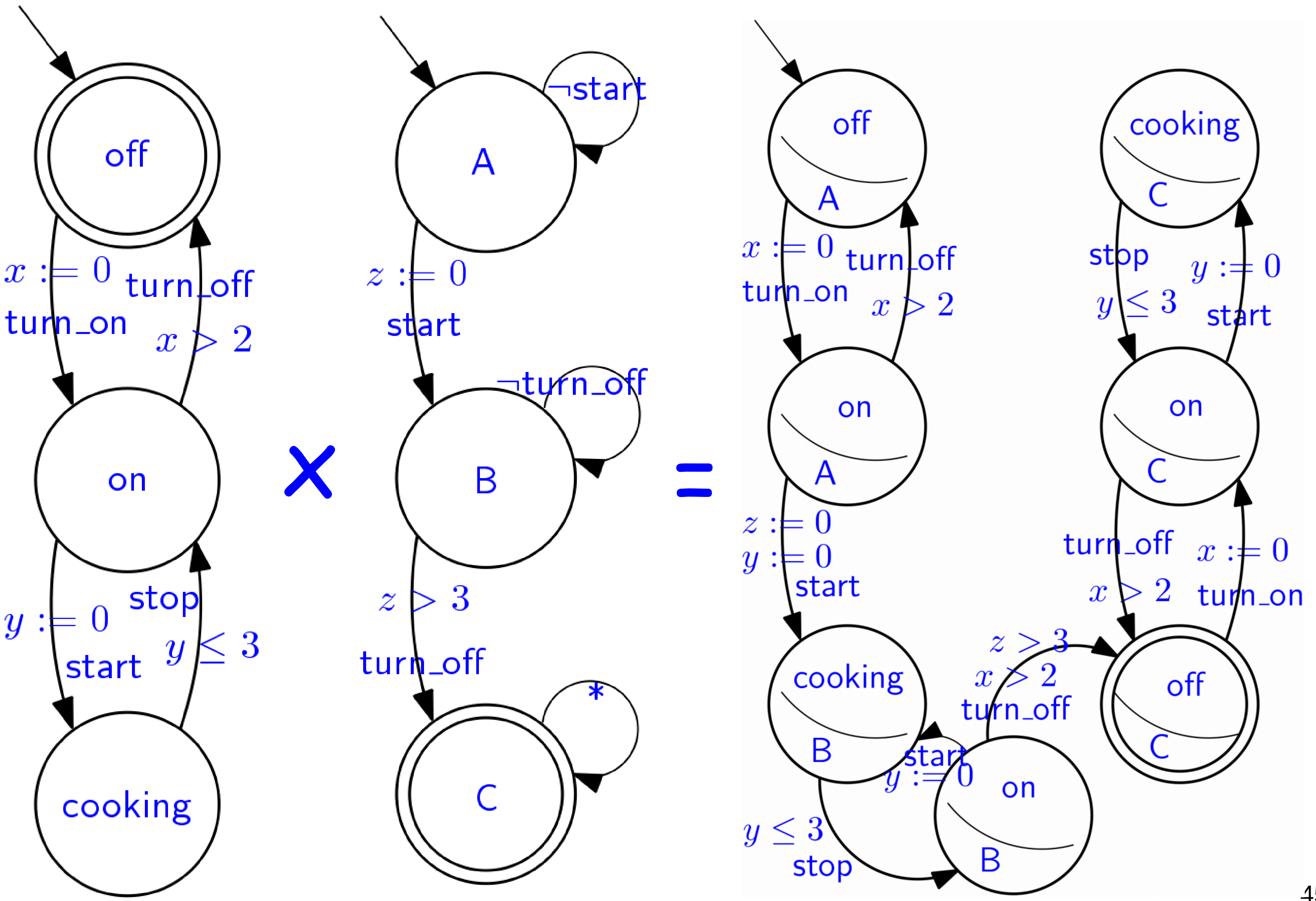
Intersection of Timed Automata

TA-Intersection: running TAs in parallel

Given TAs A, B it is always possible to build automatically a TA C that accepts precisely the words that both A and B accept.

TA C represents all possible parallel runs of A and B where a timed word is accepted if and only if both A and B would accept it. The construction is called "product automaton".

TA-Intersection: Example



TA-Intersection: running TAs in parallel

Def. Given TAs $A=[\Sigma, S^A, C^A, I^A, E^A, F^A]$ and $B=[\Sigma, S^B, C^B, I^B, E^B, F^B]$
let $C \triangleq A \times B \triangleq [\Sigma, S^C, C^C, I^C, E^C, F^C]$ be defined as:

- $S^C \triangleq S^A \times S^B$
- $C^C \triangleq C^A \cup C^B$ (assuming w.l.o.g. that they are disjoint sets)
- $I^C \triangleq \{ (s, t) \mid s \in I^A$ and $t \in I^B \}$
- $[(s, t), \sigma, c^A \wedge c^B, \rho^A \cup \rho^B, (s', t')] \in E^C$ iff
  $[s, \sigma, c^A, \rho^A, s'] \in E^A$ and $[t, \sigma, c^B, \rho^B, t'] \in E^B$
- $F^C \triangleq \{ (s, t) \mid s \in F^A$ and $t \in F^B \}$

Theorem.

$$\langle A \times B \rangle = \langle A \rangle \cap \langle B \rangle$$

Dense Real-time Model-Checking

Checking the Emptiness of Timed Automata

TA-Emptiness

Given a TA *A* it is always possible to check automatically if it accepts some timed word.

Outline of the algorithm:

- Assume that clock constraints involve integer constants only
- Define an equivalence relation over extended states (location + clocks)
- All extended states in the same equivalence class are equivalent w.r.t. satisfaction of clock constraints
  - The equivalence relation is such that there is a finite number of equivalence classes for any given TA
- Given a TA *A*, build an FSA reg(*A*) – the "region automaton":
  - the states of reg(*A*) represent the equivalence classes of the extended states of any run of of *A*
  - the edges of reg(*A*) represent evolution of any extended state within the equivalence class over any run of *A*
- Checking the emptiness of reg(*A*) is equivalent to checking *A*'s emptiness

Integer vs. Rational vs. Irrational

The domain for time is $R_{\geq 0}$

What about the domain for time constraints?

- constants in clock constraints of TAs (e.g.: $x < k$)

1. Same as the domain for time: $R_{\geq 0}$
   - e.g.: $x < \pi$
   - emptiness becomes undecidable!
2. Discrete time domain: integers Z
   - e.g.: $x < 5$
   - emptiness fully decidable (see algorithm next)
3. Dense but not continuous: rationals $Q_{\geq 0}$
   - e.g.: $x < 1/3$
   - emptiness is reducible to the integer case

Integer vs. Rational

Dense but not continuous: rationals $Q_{\geq 0}$

- Let A be a TA with rational constants
  - let m be the least common multiple of denominators of all constants appearing in the clock constraints of A
  - let A*m be the TA obtained from A by multiplying every constants in the clock constraints by m
    - A*m has only integers constants in its clock constraints
- A accepts any timed word

  $[\sigma(1), t(1)]\ [\sigma(2), t(2)] \ldots [\sigma(n), t(n)]$

  iff A*m accepts the "scaled" timed word

  $[\sigma(1), m*t(1)]\ [\sigma(2), m*t(2)] \ldots [\sigma(n), m*t(n)]$
- Hence checking the emptiness of A*m is equivalent to checking the emptiness of A

Equivalence Relation over Extended States

Let us fix a TA $A = [\Sigma, S, C, I, E, F]$ with $C = [x(1), \dots, x(n)]$

- For any clock x(i) in C let M(i) be the largest constant involving clock x(i) in any clock constraint in E
- Let $[v(1), \dots, v(n)] \in R_{\geq 0}^{n}$ denote a "clock evaluation" representing any assignment of values to clocks
- Equivalence of two clock evaluations:
  $[v(1), \dots, v(n)] \sim [v'(1), \dots, v'(n)]$ iff all of the following hold:

1. For all $1 \leq i \leq n$: $int(v(i)) = int(v'(i))$ or $v(i), v'(i) > M(i)$
2. For all $1 \leq i,j \leq n$ such that $v(i) \leq M(i)$ and $v(j) \leq M(j)$:
   $frac(v(i)) \leq frac(v(j))$ iff $frac(v'(i)) \leq frac(v'(j))$
3. For all $1 \leq i \leq n$ such that $v(i) \leq M(i)$:
   $frac(v(i)) = 0$ iff $frac(v'(i)) = 0$

Note: int(x) is the integer part of x;
frac(x) is the fractional part of x

Clock Regions

Def. A clock region is an equivalence class of clock evaluations induced by the equivalence relation ~

- For a clock evaluation $v = [v(1), \ldots, v(n)] \in R_{\geq 0}^n$, [[v]] denotes the clock region v belongs to
- As a consequence of the definition of ~, any clock region can be uniquely characterized by a finite set of constraints on clocks
  - v = [0.4; 0.9; 0.7; 0] for 4 clocks w, x, y, z
  - [[v]] is $z = 0 < w < y < x < 1$
- Fact: clock regions are always in finite number

Clock Regions (cont'd)

More systematically:

- given a set of clocks $C = [x(1), \ldots, x(n)]$
- with $M(i)$ the largest constant appearing in constraints on clock $x(i)$

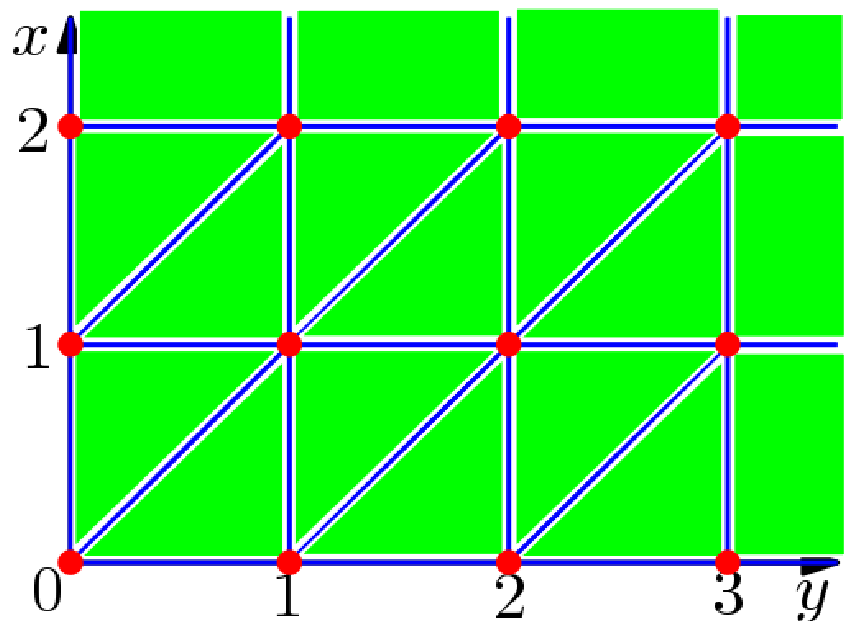
a clock region is uniquely characterized by

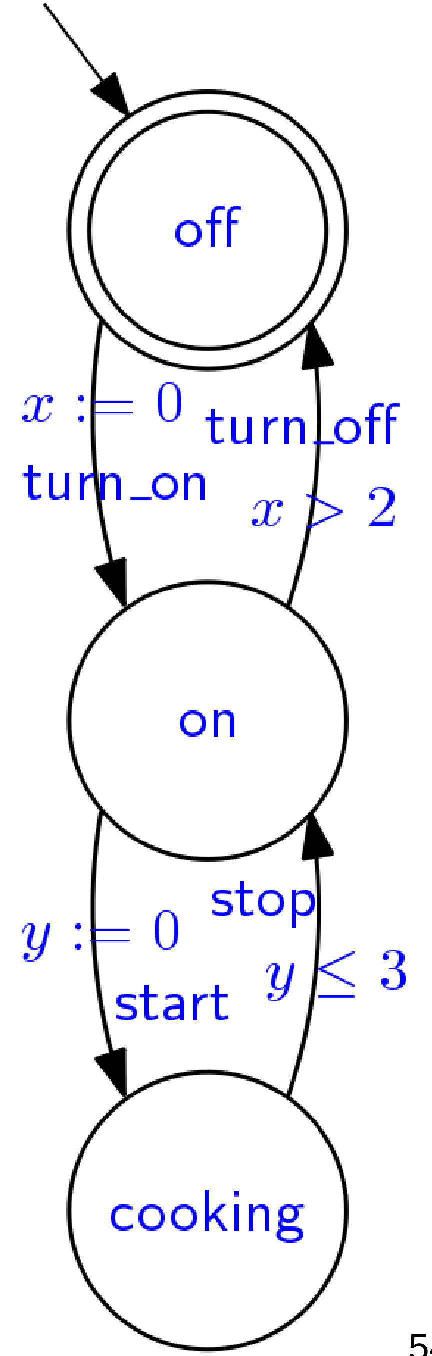
- For each clock $x(i)$ a constraint in the form:
  - $x(i) = c$ with $c = 0, 1, \ldots, M(i)$; or
  - $c - 1 < x(i) < c$ with $c = 1, \ldots, M(i)$; or
  - $x(i) > M(i)$
- For each pair of clocks $x(i)$, $x(j)$ a constraint in the form
  - $frac(x(i)) < frac(x(j))$
  - $frac(x(i)) = frac(x(j))$
  - $frac(x(i)) > frac(x(j))$

(These are unnecessary if $x(i) = c$, $x(j) = c$, $x(i) > M(i)$, or $x(j) > M(j)$)

Clock Regions: Example

- Clocks C = [x, y]
- M(x) = 2; M(y) = 3
- All 60 possible clock regions:
  - 12 corner points
  - 30 open line segments
  - 18 open regions





Time-successors of Regions

Fact: a clock evaluation v satisfies a clock constraint c iff every other clock evaluation in [[v]] satisfies c

Hence, we can say that a "clock region satisfies a clock constraint"

Def. The time successor time-succ(R) of a clock region R is the set of all clock regions (including R itself) that can be reached from R by letting time pass (i.e., without resetting any clock).

Given a clock region R it is always possible to compute all other clock regions that can be reached from R by letting time pass (i.e., without resetting any clock)

Graphically:

the time-successors of a region R are the regions that can be reached by moving along a line parallel to the diagonal in the upward direction, starting from any point in R

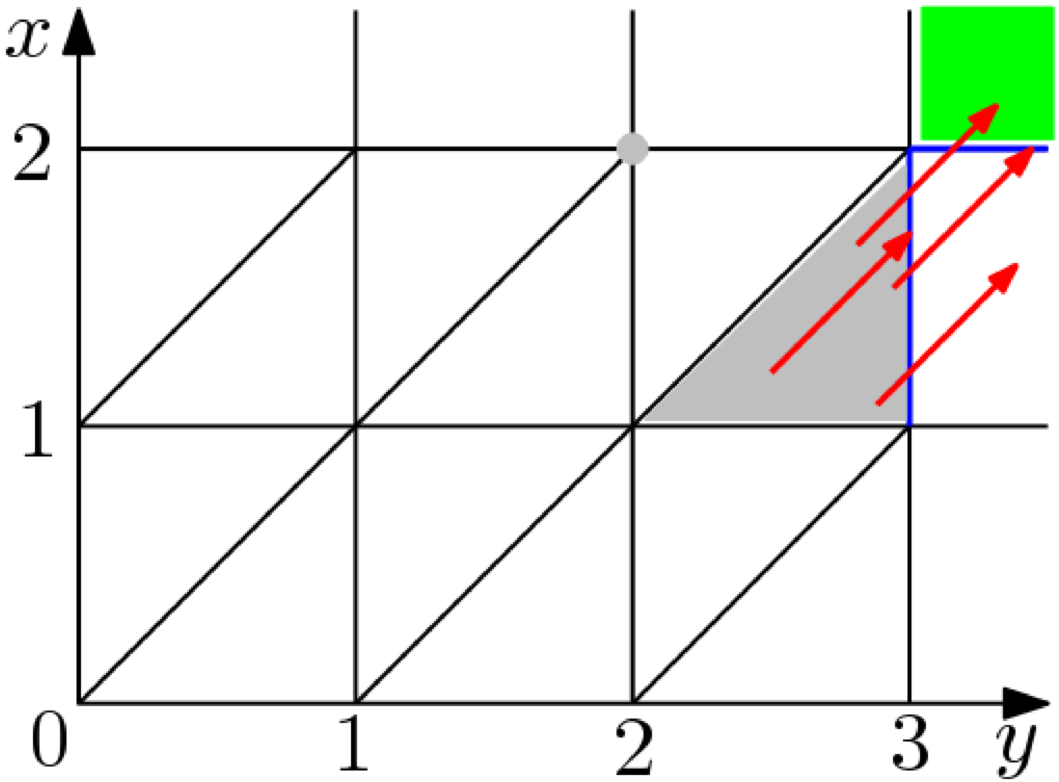
(For a precise definition see e.g.: Alur & Dill, 1994)

Time-successors of Regions: Example

Graphically: the time-successors of a region R are the regions that can be reached by moving along a line parallel to the diagonal in the upward direction, starting from any point in R

Example:

- successors of region
  $2 < y < 3;\ 1 < x < y-1$
  (other than the region itself):
  - $y > 3;\ 1 < x < 2$
  - $y > 3;\ x = 2$
  - $y = 3;\ 1 < x < 2$
  - $y > 3;\ x > 2$
- successors of region
  $y = 2;\ x = 2$ (other than the region itself):
  - $2 < y < 3;\ x > 2$
  - ...



Region Automaton Construction

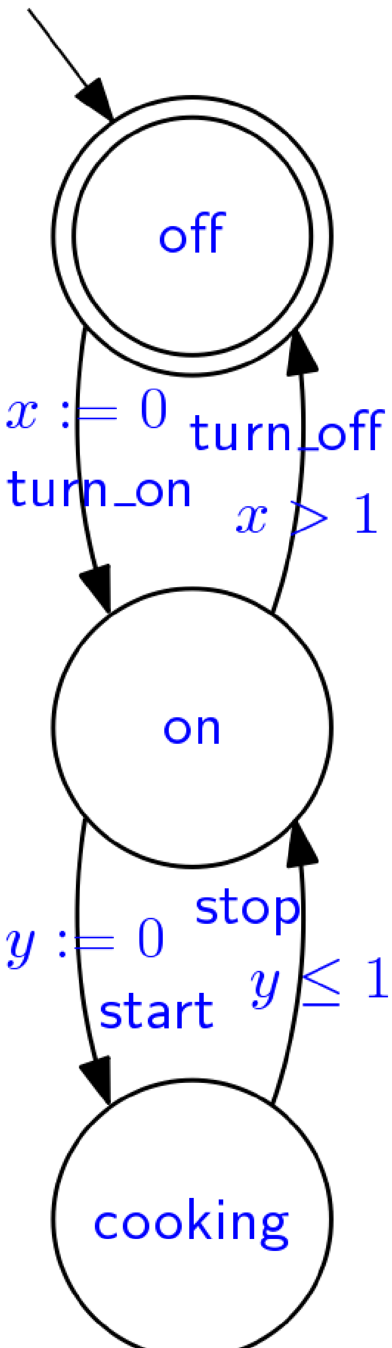
For a timed automaton A it is always possible to build an FSA $reg(A)$ (the "region automaton" of A) such that:

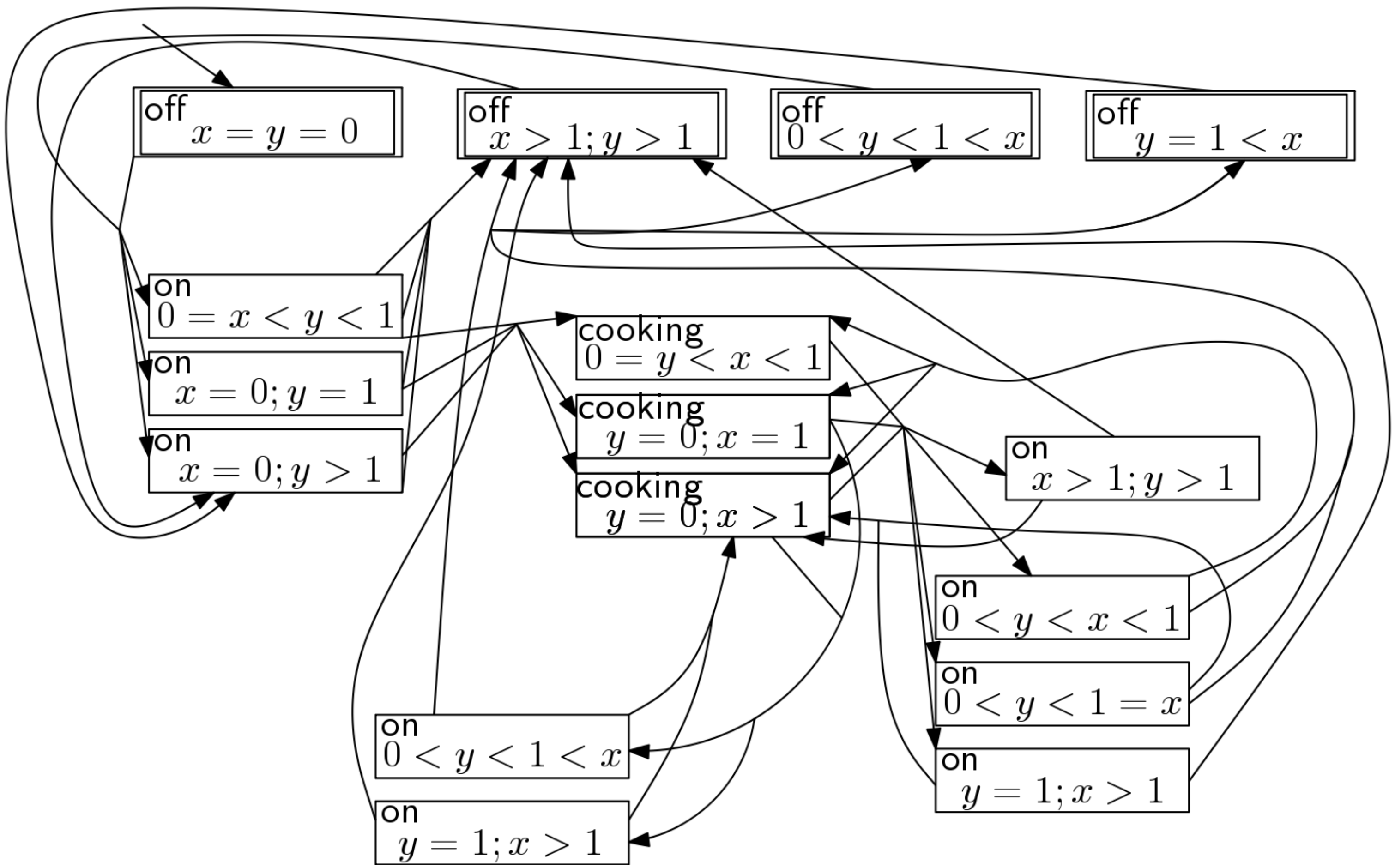
$$\langle A \rangle = \emptyset \quad \text{iff} \quad \langle reg(A) \rangle = \emptyset$$

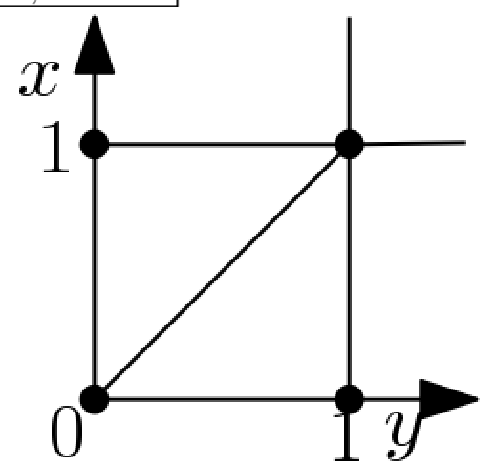
Def. Given a TA $A = [\Sigma, S, C, I, E, F]$ its region automaton $reg(A) \triangleq [\Sigma, rS, rI, rE, rF]$ is defined as:

- $rS \triangleq \{ (s, r) \mid s \in S$ and r is a clock region $\}$
- $rI \triangleq \{ (s, [[0, 0, \ldots, 0]]) \mid s \in I \}$
  - the clock region where all clocks are reset to 0
- $rE(\sigma, [s, r]) \triangleq \{ (s', r') \mid [s, \sigma, c, \rho, s'] \in E$
  and there exists a region $r'' \in$ time-succ(r)
  such that r'' satisfies c, and r' is obtained
  from r'' by resetting all clocks in ρ to 0 $\}$
- $rF \triangleq \{ (s, r) \mid s \in F \}$

Region Automaton: Example







Dense Real-time Model-Checking

Complexity, Variants, and Tools

Complexity of Emptiness Checking for TAs

- Building the region automaton and checking its emptiness takes time exponential in the size of the clock constraints
- Checking emptiness of a TA is a PSPACE-complete problem
  - Hence the region-automaton algorithm is worst-case optimal
- However, variants of the emptiness checking algorithm can achieve better performances in practice
  - mostly by using ad hoc data structures and symbolic representations of regions that can be manipulated efficiently

Variants of TA Emptiness Checking

Variants of the Emptiness Checking Algorithm are typically based on more efficient (on average) representations of regions

- Representatives
  - a clock region is represented by a concrete extended state that belongs to it
  - the concrete state is a "representative" of the region
  - if it is suitably chosen, manipulating it is equivalent to manipulating the whole region
- Clock constraints (a.k.a. zones)
  - a region is represented symbolically as a Boolean combination of clock constraints
  - successors are computed symbolically directly on the Boolean expression
- Other equivalence relations (e.g., bisimulation)
  - they can produce fewer equivalence classes

Tools for the Analysis of TAs

- Uppaal (Larsen, Petterson, Yi et al., ~1995)
- Kronos (Tripakis, Yovine et al., ~1995)
- HyTech (Henzinger et al., ~1994)
- PHAVer (Frehse, ~2005)

Remark: emptiness checking is also called
"reachability analysis"

the language of a TA *A* is empty iff the accepting states of *A* cannot be reached in any computation