Concepts of Concurrent Computation

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Lecture 10: An introduction to CSP
CSP: Origin

Communicating Sequential Processes: C.A.R. Hoare


Revised with help of S. D. Brooks and A.W. Roscoe

1985 book, revised 2004

CSP purpose

Concurrency formalism
- Expresses many concurrent situations elegantly
- Influenced design of several concurrent programming languages, in particular Occam (Transputer)

Calculus
- Formally specified: laws
- Makes it possible to prove properties of systems
A trace is a sequence of events, for example
\(<\text{coin, coffee, coin, coffee}>\)

Many traces of interest are infinite, for example
\(<\text{coin, coffee, coin, coffee, ...}>\)

(Can be defined formally, e.g. by regular expressions, but such traces definition are not part of CSP; they are descriptions of CSP process properties.)

Events come from an *alphabet*. The alphabet of all possible events is written $\Sigma$ in the following.
Processes and their traces

A CSP process is characterized (although not necessarily defined fully) by the set of its traces. For example a process may have the trace set

\{\langle \rangle, \\
\langle \text{coin, coffee} \rangle, \\
\langle \text{coin, tea} \rangle \}

The special process \textit{STOP} has a trace set consisting of a single, empty trace:

\{\langle \rangle\}
Basic CSP syntax

\[ P ::= \]

\text{STOP} \quad | \quad \text{-- Does not engage in any events}
\text{a} \rightarrow \text{Q} \quad | \quad \text{-- Engages in a, then acts like Q}
\text{Q} \parallel \text{R} \quad | \quad \text{-- Internal choice}
\text{Q} \parallel \Box \text{R} \quad | \quad \text{-- External choice}
\begin{align*}
\text{Q} \parallel_{E} \text{R} & \quad | \quad \text{-- Concurrency (E: subset of alphabet)}
\text{Q} \parallel \Sigma \text{R} & \quad | \quad \text{-- Lock-step concurrency (same as Q} \parallel \text{R)}
\text{Q} \setminus E & \quad | \quad \text{-- Hiding}
\mu \text{Q} \cdot \text{f (Q)} & \quad \text{-- Recursion}
\end{align*}
Generalization of $\to$ notation

Basic:
$$a \to P$$

Generalization:
$$x: E \to P(x)$$

Accepts any event from $E$, then executes $P(x)$ where $x$ is that event

Also written
$$? x: E \to P(x)$$

Note that if $E$ is empty then $x: E \to P(x)$ is STOP for any $P$
Some laws of concurrency

1. $P \parallel Q = Q \parallel P$
2. $(P \parallel (Q \parallel R)) = ((P \parallel Q) \parallel R)$
3. $P \parallel \text{STOP} = \text{STOP}$
4. $(c \rightarrow P) \parallel (c \rightarrow Q) = (c \rightarrow (P \parallel Q))$
5. $(c \rightarrow P) \parallel (d \rightarrow Q) = \text{STOP} \quad -- \text{If } c \neq d$
6. $(x: A \rightarrow P (x)) \parallel (y: B \rightarrow Q (y)) = (z: (A \cap B) \rightarrow (P (z) \parallel Q (z)))$
Basic notions

Processes engage in events
Example of basic notation:
\[
CVM = (\text{coin} \to \text{coffee} \to \text{coin} \to \text{coffee} \to \text{STOP})
\]

Right associativity: the above is an abbreviation for
\[
CVM = (\text{coin} \to (\text{coffee} \to (\text{coin} \to (\text{coffee} \to \text{STOP})))))
\]

Trace set of \(CVM\): \{<\text{coin, coffee, coin, coffee}>\}

The events of a process are taken from its alphabet:
\[
\alpha(CVM) = \{\text{coin, coffee}\}
\]

\text{STOP} can engage in no events
Traces

\[ \text{traces } (e \rightarrow P) = \{ \langle e \rangle + s \mid s \in \text{traces } (P) \} \]
Exercises: determine traces

\[ P ::= \]

\[ \text{STOP} \quad | \quad \text{-- Does not engage in any events} \]

\[ a \rightarrow Q \quad | \quad \text{-- Engages in } a, \text{ then acts like } Q \]

\[ Q \sqcap R \quad | \quad \text{-- Internal choice} \]

\[ Q \square R \quad | \quad \text{-- External choice} \]

\[ Q || R_{E} \quad | \quad \text{-- Concurrency (} E: \text{ subset of alphabet)} \]

\[ Q || R \quad | \quad \text{-- Lock-step concurrency (same as } Q \parallel R) \]

\[ Q \setminus E \quad | \quad \text{-- Hiding} \]

\[ \mu Q \cdot f(Q) \quad \text{-- Recursion} \]
Recursion

\[ \text{CLOCK} = (\text{tick} \rightarrow \text{CLOCK}) \]

This is an abbreviation for

\[ \text{CLOCK} = \mu P \bullet (\text{tick} \rightarrow P) \]

A recursive definition is a fixpoint equation. The \( \mu \) notation denotes the fixpoint
Accepting one of a set of events; channels

Basic notation:
\[ ? \ x : A \rightarrow P(x) \]
Accepts any event from \( A \), then executes \( P(x) \) where \( x \) is that event.

Example:
\[ ? \ y : c.A \rightarrow d.y' \]
(where \( c.A \) denotes \( \{c.x \mid x \in A\} \) and \( y' \) denotes \( y \) deprived of its initial channel name, e.g. \( (c.a)' = a \))

More convenient notation for such cases involving channels:
\[ c? \ x : A \rightarrow d!x \]
A simple buffer

COPY = c? x: A → d!x → COPY
External choice

COPYBIT = (in.0 \rightarrow out.0 \rightarrow COPYBIT

\square

in.1 \rightarrow out.1 \rightarrow COPYBIT)
External choice

\[ \text{COPY1} = \text{in}? \ x: A \rightarrow \text{out1}!x \rightarrow \text{COPY1} \]

\[ \text{COPY2} = \text{in}? \ x: B \rightarrow \text{out2}!x \rightarrow \text{COPY2} \]

\[ \text{COPY3} = \text{COPY1} \boxdot \text{COPY2} \]
Consider

\[ CHM_1 = (\text{in}1f \rightarrow \text{out}50rp \rightarrow \text{out}20rp \rightarrow \text{out}20rp \rightarrow \text{out}10rp) \]
\[ CHM_2 = (\text{in}1f \rightarrow \text{out}50rp \rightarrow \text{out}50rp) \]

\[ CHM = CHM_1 \Box CHM_2 \]
Lock-step concurrency

Consider
\[ P = \exists x: A \rightarrow P' \]
\[ Q = \exists x: B \rightarrow Q' \]

Then
\[ P \parallel Q = \exists x \rightarrow \]
\[ \begin{cases} (P' \parallel Q') & \text{if } x \in A \cap B \\ \text{STOP} & \text{otherwise} \end{cases} \]

(to be generalized soon)
More examples

\[ VM C = \]
\[
(\text{in2f} \rightarrow \\
\quad ((\text{large} \rightarrow \text{VMC}) \square \\
\quad (\text{small} \rightarrow \text{out1f} \rightarrow \text{VMC}))
\]
\[
\square \\
(\text{in1f} \rightarrow \\
\quad ((\text{small} \rightarrow \text{VMC}) \square \\
\quad (\text{in1f} \rightarrow \text{large} \rightarrow \text{VMC}))
\]

\[ \text{FOOLCUST} = (\text{in2f} \rightarrow \text{large} \rightarrow \text{FOOLCUST} \square \\
\quad \text{in1f} \rightarrow \text{large} \rightarrow \text{FOOLCUST}) \]

\[ \text{FV} = \text{FOOLCUST} \parallel \text{VMC} = \\
\quad \mu P \bullet (\text{in2f} \rightarrow \text{large} \rightarrow P \square \text{in1f} \rightarrow \text{STOP}) \]
Hiding

Consider

\[ P = a \rightarrow b \rightarrow Q \]

Assuming \( Q \) does not involve \( b \), then

\[ P \setminus \{b\} = a \rightarrow Q \]

More generally:

\[ (a \rightarrow P) \setminus E = \]

\[ \begin{align*}
& P \setminus E \quad \text{if } a \in E \\
& a \rightarrow (P \setminus E) \quad \text{if } a \notin E
\end{align*} \]
Hiding introduces internal non-determinism

Consider

\[ R = (a \rightarrow P) \square (b \rightarrow Q) \]

Then

\[ R \setminus \{a, b\} = P \sqcap Q \]
Internal non-deterministic choice

\[ CH1F = (\text{in}1f \rightarrow \]
\[ (((\text{out}20\text{rp} \rightarrow \text{out}20\text{rp} \rightarrow \]
\[ \text{out}20\text{rp} \rightarrow \text{out}20\text{rp} \rightarrow \text{out}20\text{rp} \rightarrow CH1F)\]
\[ \prod \]
\[ (\text{out}50\text{rp} \rightarrow \text{out}50\text{rp} \rightarrow CH1F)) \]
Non-deterministic internal choice: another application

\[
\text{TRANSMIT} (x) = \text{in} ? x \rightarrow \text{LOSSY} (x)
\]

\[
\text{LOSSY} (x) =
\begin{align*}
\text{out}! x & \rightarrow \text{TRANSMIT} (x) \\
\Pi & \text{out}! x \rightarrow \text{LOSSY} (x) \\
\Pi & \text{TRANSMIT} (x)
\end{align*}
\]
The general concurrency operator

Consider
\[ P = ?x: A \rightarrow P' \]
\[ Q = ?x: B \rightarrow Q' \]

Then
\[ P \parallel Q = ?x \rightarrow \]
\[ \begin{array}{ll}
\text{if } x \in E \cap A \cap B & P' \parallel Q' \\
\text{if } x \in A-B-E & P' \parallel Q \\
\text{if } x \in B-A-E & P \parallel Q' \\
\text{if } x \in (A \cap B) - E & (P' \parallel Q) \cap (P \parallel Q')
\end{array} \]
Special cases of concurrency

Lock-step concurrency:

\[ P \ || \ Q \ = \ P \ || \ Q \]

Interleaving:

\[ P \ |||\ Q \ = \ P \ ||\ Q \]

\[ \emptyset \]
Lock-step concurrency (reminder)

Consider
\[ P = \ ?x: A \rightarrow P' \]
\[ Q = \ ?x: B \rightarrow Q' \]

Then
\[ P \parallel Q = \ ?x \rightarrow \]
- \( (P' \parallel Q') \) if \( x \in E \cap A \cap B \)
- STOP otherwise
Laws of non-deterministic internal choice

\[ P \land P = P \]
\[ P \land Q = Q \land P \]
\[ P \land (Q \land R) = (P \land Q) \land R \]
\[ x \to (P \land Q) = (x \to P) \land (x \to Q) \]

\[ P \lor (Q \land R) = (P \lor Q) \land (P \lor R) \]
\[ (P \land Q) \lor R = (P \lor R) \land (Q \lor R) \]

The recursion operator is not distributive; consider:

\[ P = \mu X \cdot ((a \to X) \land (b \to X)) \]
\[ Q = (\mu X \cdot (a \to X)) \land (\mu X \cdot (b \to X)) \]
Note on external choice

From previous slide:
\[ x \rightarrow (P \land Q) = (x \rightarrow P) \land (x \rightarrow Q) \]

The question was asked in class of whether a similar property also applies to external choice \( \square \)

The conjectured property is
\[ x \rightarrow (P \square Q) = (x \rightarrow P) \square (x \rightarrow Q) \]

It does not hold, since
\[ (x \rightarrow P) \square (x \rightarrow Q) = x \rightarrow (P \land Q) \]
(As a consequence of rule on next page)
General property of external choice

$(\forall x: A \rightarrow P) \land (\forall x: B \rightarrow Q) =$

$\forall x: A \cup B \rightarrow$

- $P$ if $x \in A-B$
- $Q$ if $x \in B-A$
- $P \land Q$ if $x \in A \cap B$
Traces

\[ \text{traces } (e \rightarrow P) = \{<e> + s \mid s \in \text{traces } (P)\} \]
Exercise: determine traces

P ::= 

\text{STOP} \quad | \quad -- \text{Does not engage in any events}

a \rightarrow Q \quad | \quad -- \text{Engages in } a, \text{ then acts like } Q

Q \sqcap R \quad | \quad -- \text{Internal choice}

Q \square R \quad | \quad -- \text{External choice}

Q \parallel_{E} R \quad | \quad -- \text{Concurrency (} E : \text{ subset of alphabet)}

Q \parallel R \quad | \quad -- \text{Lock-step concurrency (same as } Q \parallel_{\Sigma} R)\)

Q \setminus E \quad | \quad -- \text{Hiding}

\mu Q \cdot f(Q) \quad -- \text{Recursion}
Refinement

Process $Q$ refines (specifically, trace-refines) process $P$ if

$$\text{traces} \ (Q) \subseteq \text{traces} \ (P)$$

For example:

$$P \ \text{refines} \ P \sqcap Q$$
The trace model is not enough

The traces of and are the same:

\[
\text{traces (P  □  Q)} = \text{traces (P)} \cup \text{traces (Q)}
\]
\[
\text{traces (P  ▷  Q)} = \text{traces (P)} \cup \text{traces (Q)}
\]

But the processes can behave differently if for example:

\[
P = a \rightarrow b \rightarrow \text{STOP}
\]
\[
Q = b \rightarrow a \rightarrow \text{STOP}
\]

Traces define what a process may do, not what it may refuse to do
Refusals

For a process $P$ and a trace $t$ of $P$:

- An event set $es \in P(\Sigma)$ is a refusal set if $P$ can forever refuse all events in $es$
- $\text{Refusals}(P)$ is the set of $P$'s refusal sets
- Convention: keep only maximal refusal sets
  (if $X$ is a refusal set and $Y \subseteq X$, then $Y$ is a refusal set)

This also leads to a notion of “failure”:

- $\text{Failures}(P, t)$ is $\text{Refusals}(P / t)$

where $P/t$ is $P$ after $t$:

$$\text{traces}(P / t) = \{u \mid t + u \in \text{traces}(P)\}$$
Comparing failures

Compare

- $P = a \rightarrow \text{STOP} \quad \Box \quad b \rightarrow \text{STOP}$
- $Q = a \rightarrow \text{STOP} \quad \mathbin{\text{Π}} \quad b \rightarrow \text{STOP}$

Same traces, but:

- Refusals (P) = $\emptyset$
- Refusals (Q) = $\{\{a\}, \{b\}\}$
Refusal sets (from labeled transition diagram)

\[ \Sigma = \{ \text{a, b, c} \} \]

\[ \{a, c\}, \{b, c\} \]

\[ \{a, c\} \]

\[ \{\Sigma\} \]

\[ a \rightarrow \text{STOP} \quad \Box \quad b \rightarrow \text{STOP} \]

\[ a \rightarrow \text{STOP} \cap b \rightarrow \text{STOP} \]

\[ a \rightarrow \text{STOP} \tau \tau b \rightarrow \text{STOP} \]

\[ \{\Sigma\} \]

\[ \{\Sigma\} \]

\[ \{\Sigma\} \]

\[ \{\Sigma\} \]
A more complete notion of refinement

Process $Q$ *failures-refines* process $P$ if both

$$\text{traces } (Q) \subseteq \text{traces } (P)$$
$$\text{failures } (Q) \subseteq \text{failures } (P)$$

Makes it possible to distinguish between $\square$ and $\square$
Divergence

A process diverges if it is not refusing all events but not communicating with the environment.

This happens if a process can engage in an infinite sequence of $\tau$ transitions.

An example of diverging process:

$$(\mu p.a \rightarrow p) \setminus a$$
The divergence model (Brookes, Roscoe)

CSP semantics is often expressed through a failures set
A failure is of the form

\[ [s, X] \]

where \( s \) is a trace (sequence of events) and \( X \) a finite set of events

A failure set must satisfy the following properties:

- \([\langle \rangle, \emptyset] \in F\]
- \([s + t, \emptyset] \in F \Rightarrow [s, \emptyset] \in F\]
- \([s, X] \in F \land \ Y \subseteq X \Rightarrow [s, Y] \in F\]
- \([s, X] \in F \land [s + \langle c\rangle, \emptyset] \notin F \Rightarrow [s, X \cup \{c\}] \in F\]
Basic CSP syntax

\[ P ::= \]

| STOP         | -- Does not engage in any events |
| a \rightarrow Q | -- Engages in a, then acts like Q |
| Q \cap R     | -- Internal choice               |
| Q \square R  | -- External choice               |
| Q \parallel R| -- Concurrency (E: subset of alphabet) |
| Q \parallel_{E} R | -- Lock-step concurrency (same as Q \parallel_{\Sigma} R) |
| Q \setminus E | -- Hiding                        |
| \mu Q \cdot f (Q) | -- Recursion                    |
CSP laws in the divergence model (1/2)

\[
\begin{align*}
P \square P & \equiv_M P \\
P \square Q & \equiv_M Q \square P \\
P \square (Q \square R) & \equiv_M (P \square Q) \square R \\
P \square (Q \land R) & \equiv_M (P \square Q) \land (P \square R) \\
P \land (Q \square R) & \equiv_M (P \land Q) \square (P \land R) \\
P \square \text{STOP} & \equiv_M P \\
(a \to (P \land Q)) & \equiv_M (a \to P) \land (a \to Q) \\
(a \to P) \square (a \to Q) & \equiv_M (a \to P) \land (a \to Q) \\
P \land P & \equiv_M P \\
P \land Q & \equiv_M Q \land P \\
P \land (Q \land R) & \equiv_M (P \land Q) \land R \\
P \parallel Q & \equiv_M Q \parallel P \\
P \parallel (Q \parallel R) & \equiv_M (P \parallel Q) \parallel R \\
P \parallel (Q \parallel R) & \equiv_M (P \parallel Q) \parallel (P \parallel R) \\
(a \to P) \parallel (b \to Q) & \equiv_M \text{STOP} \quad \text{if } a \neq b \\
& \equiv_M (a \to (P \parallel Q)) \quad \text{if } a = b \\
P \parallel \text{STOP} & \equiv_M \text{STOP}
\end{align*}
\]

(From: Brooks & Roscoe 85)
CSP laws (2/2)

\[ P || Q \equiv_M Q || P \]
\[ (P || Q) || R \equiv_M P || (Q || R) \]
\[ P || (Q \cap R) \equiv_M (P || Q) \cap (P || R) \]
\[ (a \to P) || (b \to Q) \equiv_M (a \to (P || (b \to Q))) \square (b \to ((a \to P) || Q)) \]
\[ P; (Q; R) \equiv_M (P; Q); R \]
\[ \text{STOP} || Q \equiv_M Q \]
\[ \text{SKIP}; Q \equiv_M Q \]
\[ \text{STOP}; Q \equiv_M \text{STOP} \]
\[ P; (Q \cap R) \equiv_M (P; Q) \cap (P; R) \]
\[ (P \cap Q); R \equiv_M (P; R) \cap (Q; R) \]
\[ (a \to P); Q \equiv_M (a \to P; Q) \quad \text{if } a \neq \checkmark \]
\[ (P \setminus a) \setminus b \equiv_M (P \setminus b) \setminus a \]
\[ (P \setminus a) \setminus a \equiv_M P \setminus a \]
\[ (a \to P) \setminus b \equiv_M (a \to P \setminus b) \quad \text{if } a \neq b \]
\[ \equiv_M P \setminus b \quad \text{if } a = b \]
\[ (P \cap Q) \setminus a \equiv_M (P \setminus a) \cap (Q \setminus a) \]
Some extensions

Non-timed:
- The ✔ event (not in Σ): successful termination
- Skip: successfully terminates
- Sequential composition: P ; Q
- ⊥: diverging process

Timed:
- P ⪯ Q: interrupt
- P ⪰ Q: timeout
- a ↓ P: communicate immediately
- WAIT †: same as STOP ⪰ SKIP
Example (Ouaknine)

\[ V1 = \text{coin.in} \rightarrow (\text{coke} \rightarrow V1) \square (\text{fanta} \rightarrow V1) \uparrow (\text{coin.out} \rightarrow V1) \]
Some laws no longer hold

\[ P \parallel \text{STOP} = \text{STOP} \text{ if } P \not= \bot \]
\[ \bot \parallel \text{STOP} = \bot \]

\[ (a \rightarrow P) \setminus b = a \rightarrow (P \setminus b) \text{ if } a \not= b \]
\[ (a \rightarrow P) \setminus a = P \setminus a \]
CSP: Summary

A calculus based on mathematical laws

Provides a general model of computation based on communication

Serves both as specification of concurrent systems and as a guide to implementation

One of the most influential models for concurrency work
CSP: Origin

Communicating Sequential Processes: C.A.R. Hoare

1978 paper, based in part on ideas of E.W. Dijkstra
(guarded commands, 1978 paper and “A Discipline of Programming” book)

Revised with help of S. D. Brooks and A.W. Roscoe

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(used extensively in the present slides)
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(Can be defined formally, e.g. by regular expressions, but such traces definition are not part of CSP; they are descriptions of CSP process properties.)

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The special process \textit{STOP} has a trace set consisting of a single, empty trace:

\[
\{<>\}
\]
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\[ \text{STOP} \mid \text{Does not engage in any events} \]

\[ a \rightarrow Q \mid \text{Engages in } a, \text{ then acts like } Q \]

\[ Q \sqcap R \mid \text{Internal choice} \]

\[ Q \sqcup R \mid \text{External choice} \]

\[ Q \parallel^E R \mid \text{Concurrency (} E : \text{ subset of alphabet)} \]

\[ Q \parallel^\Sigma R \mid \text{Lock-step concurrency (same as } Q \parallel R) \]

\[ Q \setminus E \mid \text{Hiding} \]

\[ \mu Q \cdot f(Q) \mid \text{Recursion} \]
**Generalization of → notation**

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**Generalization:**

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5. $(c \rightarrow P) \parallel (d \rightarrow Q) = \text{STOP} \quad \text{-- If } c \neq d$
6. $(x: A \rightarrow P(x)) \parallel (y: B \rightarrow Q(y)) = \quad (z: (A \cap B) \rightarrow (P(z) \parallel Q(z)))$
Processes engage in events

Example of basic notation:

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Right associativity: the above is an abbreviation for

\[ CVM = (\text{coin} \rightarrow (\text{coffee} \rightarrow (\text{coin} \rightarrow (\text{coffee} \rightarrow \text{STOP})))) \]

Trace set of \( CVM \): \{<\text{coin, coffee, coin, coffee}>\}

The events of a process are taken from its alphabet:

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\text{STOP} can engage in no events
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\[ Q \boxplus R \quad | \quad \text{-- External choice} \]

\[ Q \parallel_R E \quad | \quad \text{-- Concurrency (} E \text{: subset of alphabet)} \]

\[ Q \parallel_R \sum R \quad | \quad \text{-- Lock-step concurrency (same as } Q \parallel_R \sum R) \]

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Recursion

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(where \( c.A \) denotes \( \{c.x \mid x \in A\} \) and \( y' \) denotes \( y \) deprived of its initial channel name, e.g. \((c.a)' = a\))

More convenient notation for such cases involving channels:

\[ c? x: A \rightarrow d!x \]
A simple buffer

\[ \text{COPY} = c? \ x: A \rightarrow \ d!x \rightarrow \ \text{COPY} \]
**External choice**

\[ \text{COPYBIT} = (\text{in.0} \rightarrow \text{out.0} \rightarrow \text{COPYBIT} \]

\[ \quad \square \]

\[ \text{in.1} \rightarrow \text{out.1} \rightarrow \text{COPYBIT}) \]
External choice

\[ \text{COPY1} = \text{in}\, x : A \rightarrow \text{out1}\!x \rightarrow \text{COPY1} \]

\[ \text{COPY2} = \text{in}\, x : B \rightarrow \text{out2}\!x \rightarrow \text{COPY2} \]

\[ \text{COPY3} = \text{COPY1} \circledast \text{COPY2} \]
Consider

\[ CHM_1 = (\text{in}1f \rightarrow \text{out}50\text{rp} \rightarrow \text{out}20\text{rp} \rightarrow \text{out}20\text{rp} \rightarrow \text{out}10\text{rp}) \]

\[ CHM_2 = (\text{in}1f \rightarrow \text{out}50\text{rp} \rightarrow \text{out}50\text{rp}) \]

\[ CHM = CHM_1 \Box CHM_2 \]
Lock-step concurrency

Consider
\[ P = ?x : A \rightarrow P' \]
\[ Q = ?x : B \rightarrow Q' \]

Then
\[ P \parallel Q = ?x : A \cap B \rightarrow (P' \parallel Q') \]

Note that \( P \parallel Q \) is STOP for any event \( x \notin A \cap B \)

(to be generalized soon)
More examples

\[ VMC = \]
\[ (in2f \rightarrow \]
\[ ((large \rightarrow VMC) □ \]
\[ (small \rightarrow out1f \rightarrow VMC)) \]
\[ □ \]
\[ (in1f \rightarrow \]
\[ ((small \rightarrow VMC) □ \]
\[ (in1f \rightarrow large \rightarrow VMC)) \]

\[ FOOLCUST = (in2f \rightarrow large \rightarrow FOOLCUST □ \]
\[ in1f \rightarrow large \rightarrow FOOLCUST) \]

\[ FV = FOOLCUST \parallel VMC = \]
\[ µP • (in2f \rightarrow large \rightarrow FV □ \]
\[ in1f \rightarrow STOP) \]
Hiding

Consider

\[ P = a \rightarrow b \rightarrow Q \]

Assuming \( Q \) does not involve \( b \), then

\[ P \setminus \{b\} = a \rightarrow Q \]

More generally:

\[(a \rightarrow P) \setminus E = \]

\[ P \setminus E \quad \text{if} \ a \in E \]

\[ a \rightarrow (P \setminus E) \quad \text{if} \ a \notin E \]
Hiding introduces internal non-determinism

Consider

\[ R = (a \rightarrow P) \Box (b \rightarrow Q) \]

Then

\[ R \setminus \{a, b\} = P \cap Q \]
Internal non-deterministic choice

\[ CH1F = (\text{in1f} \rightarrow \]
\[ \left( (\text{out20rp} \rightarrow \text{out20rp} \rightarrow \right) \]
\[ \left( \text{out20rp} \rightarrow \text{out20rp} \rightarrow \right) \]
\[ \left( \text{out20rp} \rightarrow \text{out20rp} \rightarrow \right) \]
\[ \left( \text{out50rp} \rightarrow \text{out50rp} \rightarrow \right) \]
\[ \left( \text{CH1F} \right) \]
\[ \prod \]
\[ \left( \text{out50rp} \rightarrow \text{out50rp} \rightarrow \right) \]
\[ \left( \text{CH1F} \right) \]
Non-deterministic internal choice: another application

\[
\text{TRANSMIT}(x) = \text{in?}x \rightarrow \text{LOSSY}(x)
\]

\[
\text{LOSSY}(x) = \begin{array}{l}
\text{out!}x \rightarrow \text{TRANSMIT}(x) \\
\prod \text{out!}x \rightarrow \text{LOSSY}(x) \\
\prod \text{TRANSMIT}(x)
\end{array}
\]
The general concurrency operator

Consider
\[
P = \exists x: A \rightarrow P' \\
Q = \exists x: B \rightarrow Q'
\]

Then
\[
P \parallel Q = \exists x \rightarrow \\
\begin{align*}
\forall x \in E \cap A \cap B & : P' \parallel Q' \\
\forall x \in A \setminus B \setminus E & : P' \parallel Q \\
\forall x \in B \setminus A \setminus E & : P \parallel Q' \\
\forall x \in (A \cap B) \setminus E & : (P' \parallel Q) \cap (P \parallel Q')
\end{align*}
\]
Special cases of concurrency

Lock-step concurrency:

\[ P \parallel Q = \sum P \parallel Q \]

Interleaving:

\[ P \parallel\parallel Q = \emptyset P \parallel Q \]
Lock-step concurrency (reminder)

Consider
\[ P = \exists x: A \rightarrow P' \]
\[ Q = \exists x: B \rightarrow Q' \]

Then
\[ P \parallel Q = \exists x: A \cap B \rightarrow (P' \parallel Q') \]

Note that \( P \parallel Q \) is STOP for any event \( x \notin A \cap B \)

(to be generalized soon)
Laws of non-deterministic internal choice

\[ P \sqcap P = P \]
\[ P \sqcap Q = Q \sqcap P \]
\[ P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R \]
\[ x \rightarrow (P \sqcap Q) = (x \rightarrow P) \sqcap (x \rightarrow Q) \]

\[ P \parallel (Q \sqcap R) = (P \parallel Q) \sqcap (P \parallel R) \]
\[ (P \sqcap Q) \parallel R = (P \parallel R) \sqcap (Q \parallel R) \]

The recursion operator is not distributive; consider:

\[ P = \mu X \cdot ((a \rightarrow X) \sqcap (b \rightarrow X)) \]
\[ Q = (\mu X \cdot (a \rightarrow X)) \sqcap (\mu X \cdot (b \rightarrow X)) \]
Note on external choice

From previous slide:

\[ x \rightarrow (P \sqcap Q) = (x \rightarrow P) \sqcap (x \rightarrow Q) \]

The question was asked in class of whether a similar property also applies to external choice. □

The conjectured property is

\[ x \rightarrow (P \Box Q) = (x \rightarrow P) \Box (x \rightarrow Q) \]

It does not hold, since

\[ (x \rightarrow P) \Box (x \rightarrow Q) = x \rightarrow (P \sqcap Q) \]

(As a consequence of rule on next page)
General property of external choice

$(?x: A \rightarrow P) \mathcal{O} (?x: B \rightarrow Q) =$

$\forall x: A \cup B \rightarrow$

- $P$ if $x \in A - B$
- $Q$ if $x \in B - A$
- $P \land Q$ if $x \in A \cap B$
Traces

\[
\text{traces } (e \rightarrow P) = \{<e> + s \mid s \in \text{traces } (P)\}
\]
Exercise: determine traces

\[ P ::= \]

\[ \text{STOP} \quad \mid \quad \text{-- Does not engage in any events} \]

\[ a \rightarrow Q \quad \mid \quad \text{-- Engages in } a, \text{ then acts like } Q \]

\[ Q \sqcap R \quad \mid \quad \text{-- Internal choice} \]

\[ Q \Box R \quad \mid \quad \text{-- External choice} \]

\[ Q \sqcup R \quad \mid \quad \text{-- Concurrency (} E \text{: subset of alphabet)} \]

\[ Q \sqcup R \quad \mid \quad \text{-- Lock-step concurrency (same as } Q \sqcup R \text{)} \]

\[ Q \setminus E \quad \mid \quad \text{-- Hiding} \]

\[ \mu Q \cdot f(Q) \quad \text{-- Recursion} \]
Refinement

Process $Q$ refines (specifically, trace-refines) process $P$ if

$$\text{traces (Q)} \subseteq \text{traces (P)}$$

For example:

$P$ refines $P \cap Q$
The trace model is not enough

The traces of and are the same:

\[
\text{traces (P } \square \text{ Q)} = \text{traces (P)} \cup \text{traces (Q)}
\]
\[
\text{traces (P } \sqcap \text{ Q)} = \text{traces (P)} \cup \text{traces (Q)}
\]

But the processes can behave differently if for example:

\[
P = a \rightarrow b \rightarrow \text{STOP}
\]
\[
Q = b \rightarrow a \rightarrow \text{STOP}
\]

Traces define what a process may do, not what it may refuse to do
Refusals

For a process \( P \) and a trace \( t \) of \( P \):

- An event set \( es \in P (\sum) \) is a *refusal set* if \( P \) can forever refuse all events in \( es \)
- \( \text{Refusals} (P) \) is the set of \( P \)'s refusal sets
- Convention: keep only maximal refusal sets
  (if \( X \) is a refusal set and \( Y \subseteq X \), then \( Y \) is a refusal set)

This also leads to a notion of “failure“:

- \( \text{Failures} (P, t) \) is \( \text{Refusals} (P / t) \)

where \( P / t \) is \( \text{P after } t \):

\[
\text{traces} (P / t) = \{ u \mid t + u \in \text{traces} (P) \}
\]
Comparing failures

Compare

- \( P = a \rightarrow \text{STOP} \quad \square \quad b \rightarrow \text{STOP} \)
- \( Q = a \rightarrow \text{STOP} \quad \cap \quad b \rightarrow \text{STOP} \)

Same traces, but:

- Refusals \((P) = \emptyset\)
- Refusals \((Q) = \{\{a\}, \{b\}\}\)
Refusal sets (from labeled transition diagram)

\[ \Sigma = \{ a, b, c \} \]

\[ \{ \Sigma \} \]

\[ \{ \Sigma \} \]

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A more complete notion of refinement

Process $Q$ \textit{failures-refines} process $P$ if both

\begin{align*}
\text{traces (Q) } & \subseteq \text{traces (P)} \\
\text{failures (Q) } & \subseteq \text{failures (P)}
\end{align*}

Makes it possible to distinguish between $\Box$ and $\Pi$
Divergence

A process diverges if it is not refusing all events but not communicating with the environment.

This happens if a process can engage in an infinite sequence of $\tau$ transitions.

An example of diverging process:

$$(\mu p.a \rightarrow p) \setminus a$$
The divergence model (Brookes, Roscoe)

CSP semantics is often expressed through a failures set. A failure is of the form

\[[s, X]\]

where \(s\) is a trace (sequence of events) and \(X\) a finite set of events.

A failure set must satisfy the following properties:

- \([\langle\rangle, \emptyset] \in F\)
- \([s + t, \emptyset] \in F \implies [s, \emptyset] \in F\)
- \([s, X] \in F \land Y \subseteq X \implies [s, Y] \in F\)
- \([s, X] \in F \land [s + \langle c\rangle, \emptyset] \notin F \implies [s, X \cup \{c\}] \in F\)
Basic CSP syntax

\[ P ::= \]

\[
\text{STOP} \quad | \quad \text{-- Does not engage in any events}
\]

\[
a \rightarrow Q \quad | \quad \text{-- Engages in } a, \text{ then acts like } Q
\]

\[
Q \sqcap R \quad | \quad \text{-- Internal choice}
\]

\[
Q \lozenge R \quad | \quad \text{-- External choice}
\]

\[
Q \parallel_R E \quad | \quad \text{-- Concurrency (} E \text{: subset of alphabet)}
\]

\[
Q \parallel_R \Sigma \quad | \quad \text{-- Lock-step concurrency (same as } Q \parallel_R \Sigma)
\]

\[
Q \setminus E \quad | \quad \text{-- Hiding}
\]

\[
\mu Q \cdot f (Q) \quad \text{-- Recursion}
\]
CSP laws in the divergence model (1/2)

\[
\begin{align*}
P \Box P & \equiv_M P \\
P \Box Q & \equiv_M Q \Box P \\
P \Box (Q \Box R) & \equiv_M (P \Box Q) \Box R \\
P \Box (Q \cap R) & \equiv_M (P \Box Q) \cap (P \Box R) \\
P \cap (Q \Box R) & \equiv_M (P \cap Q) \Box (P \cap R) \\
P \Box \text{STOP} & \equiv_M P \\
(a \rightarrow (P \cap Q)) & \equiv_M (a \rightarrow P) \cap (a \rightarrow Q) \\
(a \rightarrow P) \Box (a \rightarrow Q) & \equiv_M (a \rightarrow P) \cap (a \rightarrow Q) \\
P \cap P & \equiv_M P \\
P \cap Q & \equiv_M Q \cap P \\
P \cap (Q \cap R) & \equiv_M (P \cap Q) \cap R \\
P \parallel Q & \equiv_M Q \parallel P \\
P \parallel (Q \parallel R) & \equiv_M (P \parallel Q) \parallel R \\
P \parallel (Q \cap R) & \equiv_M (P \parallel Q) \cap (P \parallel R) \\
(a \rightarrow P) \parallel (b \rightarrow Q) & \equiv_M \text{STOP} \quad \text{if } a \neq b \\
& \equiv_M (a \rightarrow (P \parallel Q)) \quad \text{if } a = b \\
P \parallel \text{STOP} & \equiv_M \text{STOP}
\end{align*}
\]

(From: Brooks & Roscoe 85)
CSP laws (2/2)

\[ P \| Q \equiv_M Q \| P \]
\[ (P \| Q) \| R \equiv_M P \| (Q \| R) \]
\[ P \| (Q \cap R) \equiv_M (P \| Q) \cap (P \| R) \]
\[ (a \rightarrow P) \| (b \rightarrow Q) \equiv_M (a \rightarrow (P \| (b \rightarrow Q))) \Box (b \rightarrow ((a \rightarrow P) \| Q)) \]
\[ P ; (Q ; R) \equiv_M (P ; Q) ; R \]
\[ STOP \| Q \equiv_M Q \]
\[ SKIP ; Q \equiv_M Q \]
\[ STOP ; Q \equiv_M STOP \]
\[ P ; (Q \cap R) \equiv_M (P ; Q) \cap (P ; R) \]
\[ (P \cap Q) ; R \equiv_M (P ; R) \cap (Q ; R) \]
\[ (a \rightarrow P) ; Q \equiv_M (a \rightarrow P ; Q) \quad \text{if } a \neq \sqrt{\} \]
\[ (P \setminus a) \setminus b \equiv_M (P \setminus b) \setminus a \]
\[ (P \setminus a) \setminus a \equiv_M P \setminus a \]
\[ (a \rightarrow P) \setminus b \equiv_M (a \rightarrow P \setminus b) \quad \text{if } a \neq b \]
\[ \equiv_M P \setminus b \quad \text{if } a = b \]
\[ (P \cap Q) \setminus a \equiv_M (P \setminus a) \cap (Q \setminus a) \]
Some extensions

Non-timed:
- The ✓ event (not in \( \Sigma \)): successful termination
- Skip: successfully terminates
- Sequential composition: \( P ; Q \)
- \( \bot \): diverging process

Timed:
- \( P \stackrel{+}{\rightarrow} Q \): interrupt
- \( P \triangleright Q \): timeout
- \( \alpha \stackrel{\downarrow}{\rightarrow} P \): communicate immediately
- \( \text{WAIT} \uparrow \): same as \( \text{STOP} \uparrow \text{SKIP} \)
Example (Ouaknine)

\[ V1 = \text{coin.in} \rightarrow ((\text{coke} \rightarrow V1) \square (\text{fanta} \rightarrow V1)) \uparrow (\text{coin.out} \rightarrow V1) \]
Some laws no longer hold

\[ P \parallel \text{STOP} = \text{STOP} \text{ if } P \neq \bot \]
\[ \bot \parallel \text{STOP} = \bot \]

\[ (a \to P) \setminus b = a \to (P \setminus b) \text{ if } a \neq b \]
\[ (a \to P) \setminus a = P \setminus a \]
CSP: Summary

A calculus based on mathematical laws

Provides a general model of computation based on communication

Serves both as specification of concurrent systems and as a guide to implementation

One of the most influential models for concurrency work