Software Verification

Assertion Inference

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The Program Verification problem:

- **Given**: a program $P$ and a specification $S = [\text{Pre}, \text{Post}]$
- **Determine**: if every execution of $P$, for any value of input parameters, satisfies $S$
- **Equivalently**: establish whether $\{\text{Pre}\} P \{\text{Post}\}$ is (totally) correct

- A *general and fully automated solution* to the Program Verification problem is unachievable because the problem is undecidable

- One of the consequences of this intrinsic limitation is the *impossibility* of computing intermediate assertions fully automatically

  (It is not an obvious consequence: formally, a reduction between undecidable problems)
Proving Programs Automatically

The Program Verification problem:

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- **Determine**: if every execution of $P$, for any value of input parameters, satisfies $S$
- **Equivalently**: establish whether $\{\text{Pre}\} P \{\text{Post}\}$ is (totally) correct

One way to put it, practically:

Proving the correctness of a computer program requires knowledge about the program that is not readily available in the program text

-- Chang & Leino

In this lecture, we survey techniques to automatically infer assertions in interesting special cases
The Assertion Inference Paradox

**Correctness** is consistency of implementation to specification

The **paradox**: if the specification is inferred from the implementation, what do we prove?
The Assertion Inference Paradox

The paradox:
if the specification is inferred from the implementation, what do we prove?

Possible retorts:
- The paradox only arises for correctness proofs; there are other applications (e.g. reverse-engineering legacy software)
- The result may be presented to a programmer for assessment
- Inferred specification may be inconsistent, thus denoting a problem
The Assertion Inference Paradox

The paradox:

if the specification is inferred from the implementation, what do we prove?

The paradox does not arise if we only infer intermediate assertions and not specifications (pre and postconditions)

- Intermediate assertions are a technical means to an end (proving correctness)
  - tools infer loop invariants
- The specification is a formal description of what the implementation should do
  - programmers write specifications
Invariants

Let us consider a general (and somewhat informal) definition of invariant:

Def. Invariant: assertion whose truth is preserved by the execution of (parts of) a program.

\[ x: \text{INTEGER} \]
from \[ x := 1 \text{ until ... loop } x := -x \text{ end} \]

Some invariants:
- \[ -1 \leq x \leq 1 \]
- \[ x = -1 \lor x = 0 \lor x = 1 \]
- \[ x \geq -10 \]
Kinds of Invariants

Def. Invariant: assertion whose truth is preserved by the execution of (parts of) a program.

We can identify different types of invariants, according to what parts of the program preserve the invariant:

- Location invariant at $x$: assertion that holds whenever the computation reaches location $x$
- Program invariant: predicate that holds in any reachable state of the computation
- Class invariant: predicate that holds between (external) feature invocations
- Loop invariant: predicate that holds after every iteration of a loop body

```
{P} A {I}
{I \land \neg c} B {I}
________________
{P}
from A until c loop B end
{I \land c}
```
Kinds of Invariants

1: \( x: \text{INTEGER} \)
2: 
3: \text{from } x := 1 \text{ until ...}
4: 
5: loop \( x := -x \) end

- Location invariant at 2:
- Loop invariant:
- Program invariant:
Kinds of Invariants

1: x: INTEGER
2: 
3: from x := 1
4: until ...
5: loop x := -x end

- Location invariant at 2:
  \[ x = 0 \]

- Loop invariant:
  \[ x = -1 \lor x = 1 \]

- Program invariant:
  \[ x \geq -10 \]
Focus on Loop Invariants

If we have loop invariants we can get (basically) everything else at little cost

- while getting loop invariants requires invention

In the following discussion we focus on loop invariants (and call them simply “invariants”)

This focus is also consistent with the Assertion Inference Paradox
Focus on Loop Invariants

The various kinds of invariants are closely related by the inference rules of Hoare logic.

- If $L_x$ is a location invariant at $x$ then:
  $$\forall x \Rightarrow L_x$$
  is a program invariant.

- If $P$ is a program invariant then it is also a location invariant at every location $x$.

- If $I$ is a loop invariant of:
  $$x: \text{from ... until } c \text{ loop ... end}$$
  then $I \land c$ is a location invariant at $x+1$.

- If $L$ is a location invariant at $x+1$:
  $$x: \ a := b + 3$$
  then $L \ [b + 3 / a]$ is a location invariant at $x$.

- Etc...

\[
\begin{array}{l}
\{P\} \ A \ \{I\} \\
\{I \land \neg \ c\} \ B \ \{I\} \\
\hline
\{P\} \\
\text{from } A \text{ until } c \\
\text{loop } B \text{ end} \\
\{I \land c\}
\end{array}
\]

\[
\{P \ [e / x]\} \ x := e \ \{P\}
\]
Classification of invariant inference techniques:

- **Dynamic** techniques
- **Static** techniques
  - statistical techniques
  - exact techniques

(Roughly) direction of increasing: soundness, completeness, mathematical sophistication
Exact Static Techniques for Invariant Inference
Static Invariant Inference: classification

Static exact techniques for invariant inference are further classified in categories:

- Direct
- Assertion-based
  - postcondition mutation
- Based on abstract interpretation
- Constraint-based
  - usually, template-based
Exact Static Techniques for Invariant Inference:

Postcondition-mutation Approach
The Role of User-provided Contracts

Techniques for invariant inference rarely take advantage of other annotations in the program text, such as contracts provided by the user.

- Not every annotation can (or should, cf. Assertion Inference Paradox) be inferred automatically.

However, there is a close connection between a loop's invariant and its postcondition.
The Role of User-provided Contracts

However, there is a close connection between a loop's invariant and its postcondition

Semantically, the invariant is a weakened form of the postcondition

- A larger set of program states

Example:

```plaintext
from x := 0 until x = n loop x := x + 1 end

• Post: x = n (with n > 0)
• Invariant: 0 ≤ x ≤ n
```
Invariants by Postcondition Mutation

- In a nutshell:
  Static verification of candidate invariants obtained by mutating postconditions

  - Assume the availability of postconditions
  - Mutate postconditions according to various heuristics
    - the heuristics mirror common patterns that link postconditions to invariants
    - each mutated postcondition is a candidate invariant
  - Verify which candidates are indeed invariants
    - With an automatic program prover such as Boogie
  - Retain all verified invariants

- 2009 - gin-pink
- 2013 - DynaMate
Loop invariant inference

- **Input**
  - Pre
  - Post
  - Routine

- **Candidate Invariants**
  - mutate
  - checking invariance

- **Loop Invariants**
  - proving correctness (possibly using additional info)

- **Output**
  - {Pre} Routine {Post}
Maximum value of an array

\[
\text{max} (A: \text{ARRAY}[T]; n: \text{INTEGER}): T \\
\text{require } A.length = n \geq 1 \\
\text{local } i: \text{INTEGER} \\
\text{do} \\
\quad \text{from } i := 0 ; \text{ Result := } A[1] ; \\
\quad \text{until } i = n \\
\text{loop} \\
\quad i := i + 1 \\
\quad \text{if } \text{ Result } \leq A[i] \text{ then } \text{ Result := } A[i] \text{ end} \\
\text{end} \\
\text{ensure } \left( \forall 1 \leq j \leq n \Rightarrow A[j] \leq \text{Result} \right) \text{ and } \left( \exists 1 \leq j \leq n \land A[j] = \text{Result} \right)
Maximum value of an array

```plaintext
max (A: ARRAY [T] ; n: INTEGER): T
    require A.length = n ≥ 1
    ensure  ( ∀ 1 ≤ j ≤ n ⇒ A[j] ≤ Result ) and
            ( ∃ 1 ≤ j ≤ n ∧ A[j] = Result )
```

- **Constant relaxation**: replace “constant” n by “variable” i
- **Term dropping**: remove second conjunct

**Invariant**: ∀ 1 ≤ j ≤ i ⇒ A[j] ≤ Result
Maximum value of an array (cont'd)

\[
\text{max} \ (A: \text{ARRAY} \ [T] ; n: \text{INTEGER}): T \\
\text{require } A.\text{length} = n \geq 1 \\
\text{ensure } ( \forall 1 \leq j \leq n \Rightarrow A[j] \leq \text{Result} ) \text{ and } \\
( \exists 1 \leq j \leq n \wedge A[j] = \text{Result} )
\]

- **Term dropping**: remove first conjunct

  Invariant: \( \exists 1 \leq j \leq n \wedge A[j] = \text{Result} \)
Maximum value of an array (2\textsuperscript{nd} version)

\textit{max\_v2} (A: ARRAY [T]; n: INTEGER): T

\begin{itemize}
  \item require \textit{A.length} = n \geq 1
  \item local \textit{i}: INTEGER
  \item do
  \begin{itemize}
    \item from \textit{i} := 1; \textit{Result} := \textit{A}[1];
    \item until \textit{i} > \textit{n}
  \end{itemize}
  \item loop
    \begin{itemize}
      \item if \textit{Result} \leq \textit{A}[\textit{i}] then \textit{Result} := \textit{A}[\textit{i}] end
    \end{itemize}
  \item \textit{i} := \textit{i} + 1
  \end{itemize}
  \item ensure \forall 1 \leq \textit{j} \leq \textit{n} \Rightarrow \textit{A}[\textit{j}] \leq \textit{Result}
\end{itemize}
Maximum value of an array (2\textsuperscript{nd} version)

```
max_v2 (A: ARRAY [T]; n: INTEGER): T
require A.length = n ≥ 1
ensure ∀ 1 ≤ j ≤ n ⇒ A[j] ≤ Result
```

- **Constant relaxation:** replace “constant” \( n \) by “variable” \( i \)
  \[
  ∀ 1 ≤ j ≤ i ⇒ A[j] ≤ Result
  \]

- **Variable aging:**
  use expression representing the previous value of \( i \): \( i - 1 \)
  Invariant: \( ∀ 1 ≤ j ≤ i - 1 ⇒ A[j] ≤ Result \)
Postcondition Mutation Heuristics

Constant relaxation

- replace “constant” by “variable”
  - cannot/may be changed by any of the loop bodies

Uncoupling

- replace subexpression appearing twice by two subexpressions
  - for example: subexpression = variable id

Term dropping

- remove a conjunct

Variable aging

- replace subexpression by another expression representing its previous value
Invariant Inference: the Algorithm

**Goal:** find invariants of loops in procedure proc

For each:

- **post**: postcondition clause of proc
- **loop**: outer loop in proc

compute all mutations $M$ of post w.r.t. loop

- considering postcondition clauses separately implements term dropping

**Result:** any formula in $M$ which can be verified as invariant of any loop in proc
**Array Partitioning**

\[
\text{partition}(A: \text{ARRAY}[T]; n: \text{INTEGER}; \text{pivot}: T): \text{INTEGER}
\]

require \(A\.\text{length} = n \geq 1\)

local \(l, h: \text{INTEGER}\)

do
\[
\text{from } l := 1 \text{ ; } h := n \text{ until } l = h
\]
loop
\[
\text{from until } l = h \text{ or } A[l] > \text{pivot} \text{ loop } l := l + 1 \text{ end}
\]
\[
\text{from until } l = h \text{ or } \text{pivot} > A[h] \text{ loop } h := h - 1 \text{ end}
\]
\[
A\.\text{swap}(l, h)
\]
end
\[
\text{if } \text{pivot} \leq A[l] \text{ then } l := l - 1 \text{ end ; } h := l ; \text{Result} := h
\]
ensure \((\forall 1 \leq k \leq \text{Result} \Rightarrow A[k] \leq \text{pivot})\) and
\[
(\forall \text{Result} < k \leq n \Rightarrow A[k] \geq \text{pivot})
\]
Array Partitioning

\[
\text{partition} \ (A: \ \text{ARRAY} \ [T]; \ n: \ \text{INTEGER}; \ \text{pivot}: \ T): \ \text{INTEGER} \\
\text{require} \ A.\text{length} = n \geq 1 \\
\text{ensure} \ (\forall 1 \leq k \leq \text{Result} \Rightarrow A[k] \leq \text{pivot}) \ \text{and} \\
(\forall \text{Result} < k \leq n \Rightarrow A[k] \geq \text{pivot})
\]

- **Uncoupling:** replace first occurrence of \text{Result} by \(l\) and second by \(h\)
  
  \( (\forall 1 \leq k \leq l \Rightarrow A[k] \leq \text{pivot}) \ \text{and} \ (\forall h < k \leq n \Rightarrow A[k] \geq \text{pivot}) \)

- **Variable aging:** use expression representing the previous value of \(l\): \(l - 1\)

**Invariant:**

\( (\forall 1 \leq k \leq l - 1 \Rightarrow A[k] \leq \text{pivot}) \ \text{and} \ (\forall h < k \leq n \Rightarrow A[k] \geq \text{pivot}) \)
Array Partitioning

\[\text{partition} \ (A: \text{ARRAY} \ [T]; \ n: \text{INTEGER}; \ \text{pivot}: \ T): \text{INTEGER}\]
\[
\text{require} \ A.\text{length} = n \geq 1
\]
\[
\text{ensure} \ \ (\forall \ 1 \leq k \leq \text{Result} \Rightarrow A[k] \leq \text{pivot}) \text{ and } (\forall \ \text{Result} < k \leq n \Rightarrow A[k] \geq \text{pivot})
\]

- **Term dropping**: remove first conjunct
  \[\forall \ \text{Result} < k \leq n \Rightarrow A[k] \geq \text{pivot}\]

- **Constant relaxation**: replace “constant” Result by “variable” h

  Invariant: \[\forall \ h < k \leq n \Rightarrow A[k] \geq \text{pivot}\]
Postcondition Mutation in DynaMate

DynaMate is a tool that combines postcondition mutation with template-based techniques for invariant inference and with automated testing. It finds loop invariants to verify Java/JML programs.

Experiments on 28 methods of java.util:

<table>
<thead>
<tr>
<th># proved methods</th>
<th>% proved proof obligations</th>
<th># invariants post mutations</th>
<th># other invariants</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>97 %</td>
<td>10</td>
<td>15</td>
<td>45 min.</td>
</tr>
</tbody>
</table>
Limitations of the approach

Some invariants are not mutations of the postcondition

- “completeness” of the postcondition
- integration with other techniques
- more heuristics

Combinatorial explosion

- predefined mutations, time out

Dependencies

- especially with nested loops
- dynamic checking

Limitations of automated reasoning techniques
Exact Static Techniques for Invariant Inference:

Constraint-based Approach
In a nutshell:

encode semantics of iteration as constraints on a template invariant

- Choose a template invariant expression
  - template defines a (infinite) set of assertions
- Encode the loop semantics as a set of constraints on the template
  - initiation + consecution
- Solve the constraints
  - this is usually the complex part
- Any solution is an invariant

E.g.

2003 -- Henny Sipma et al.
2004 -- Zohar Manna et al.
2007 -- Tom Henzinger et al.
Constraint-based Inv. Inference: Example

**dummy Routine** (n: NATURAL)
local x: NATURAL do
  from x := 0
  until x ≥ n
  loop x := x + 1 end
end

- **Template** invariant expression:
  \[ T = c \cdot x + d \cdot n + e \leq 0 \]

- **Constraints encoding loop semantics:**
  - **Initiation:** “T holds for the initial values of x and n”
    \[ T [0/x; n_0/n] \equiv c \cdot 0 + d \cdot n_0 + e \leq 0 \equiv d \cdot n_0 + e \leq 0 \]
**Constraint-based Inv. Inference: Example**

- **Constraints encoding loop semantics:**

  - **Consecution:** “if T holds and one iteration of the loop is executed, T still holds”
  
  \[ T \left[ x/x; n/n \right] \land (\neg(x \geq n) \land x' = x + 1 \land n' = n ) \Rightarrow T \left[ x'/x; n'/n \right] \]

- **Solving** the constraints requires to eliminate occurrences of \( x, x', n, n' \)
  - For linear constraints we can use **Farkas' Lemma**

```plaintext
dummy_routine (n: NATURAL)
local x: NATURAL do
  from x := 0
  until x \geq n
  loop x := x + 1 end
end
```
Farkas's Lemma (1902)

Let $S$ be a system of linear inequalities over $n$ real variables:

$$ S \triangleq \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n + b_1 & \leq & 0 \\ \vdots & & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + b_m & \leq & 0 \end{bmatrix} $$

and let $\Psi$ be a linear inequality:

$$ \psi \triangleq c_1x_1 + \cdots + c_nx_n + d \leq 0 $$

Then $S \Rightarrow \Psi$ is valid iff $S$ is unsatisfiable or there exist $m + 1$ real nonnegative coefficients $\lambda_0, \lambda_1, \ldots, \lambda_m$ such that:

$$ c_j = \sum_{i=1}^{m} \lambda_i a_{ij} \quad (1 \leq j \leq m) \quad d = -\lambda_0 + \sum_{i=1}^{m} \lambda_i b_i $$
Use Farkas' lemma to turn the consecution constraint:

\[ T[x/x; n/n] \land x < n \land x' = x + 1 \land n' = n \Rightarrow T[x'/x; n'/n] \]

into a constraint over \( c, d, \) and \( e \) only.

\[
\begin{array}{l|l l l}
\lambda_1 & cx & +dn & +e & \leq 0 \\
\lambda_2 & x & -n & +1 & \leq 0 \\
\lambda_3 & -x & +x' & -1 & \leq 0 \\
\lambda_4 & x & -x' & +1 & \leq 0 \\
\lambda_5 & -n & +n' & \leq 0 \\
\lambda_6 & n & -n' & \leq 0 \\
\hline
& cx' & +dn' & +e & \leq 0
\end{array}
\]
## Constraint-based Inv. Inference: Example

| \( \lambda_1 \) | \( cx + dn \) | +e | \( \leq 0 \) |
| \( \lambda_2 \) | \( x - n \) | +1 | \( \leq 0 \) |
| \( \lambda_3 \) | \( -x + x' \) | -1 | \( \leq 0 \) |
| \( \lambda_4 \) | \( x - x' \) | +1 | \( \leq 0 \) |
| \( \lambda_5 \) | \( -n + n' \) | \( \leq 0 \) |
| \( \lambda_6 \) | \( n - n' \) | \( \leq 0 \) |

\[ cx' + dn' + e \leq 0 \]

\( \Phi \triangleq \exists \lambda_0, \ldots, \lambda_6 \begin{bmatrix} 
\lambda_0, \ldots, \lambda_6 \geq 0 \\
\lambda_1 c + \lambda_2 - \lambda_3 + \lambda_4 = 0 \\
\lambda_1 d - \lambda_2 - \lambda_5 + \lambda_6 = 0 \\
\lambda_3 - \lambda_4 = c \\
\lambda_5 - \lambda_6 = d \\
-\lambda_0 + \lambda_1 e + \lambda_2 - \lambda_3 + \lambda_4 = e 
\end{bmatrix} \]
Constraint-based Inv. Inference: Example

\[ \Phi \triangleq \exists \lambda_0, \ldots, \lambda_6 \]
\[
\begin{bmatrix}
\lambda_0, \ldots, \lambda_6 \geq 0 \\
\lambda_1 c + \lambda_2 - \lambda_3 + \lambda_4 = 0 \\
\lambda_1 d - \lambda_2 - \lambda_5 + \lambda_6 = 0 \\
\lambda_3 - \lambda_4 = c \\
\lambda_5 - \lambda_6 = d \\
-\lambda_0 + \lambda_1 e + \lambda_2 - \lambda_3 + \lambda_4 = e
\end{bmatrix}
\]

Finally, eliminate existential quantifiers from \( \Phi \) to get the constraint:

\[ c \leq 0 \lor (c + d = 0 \land e \leq 0) \]

(Quantifier elimination is also quite technical, but there are tools that do that for us)
Any solution \([c, d, e]\) to:

- **Initiation and Consecution:**
  \[
  (d \cdot n_0 + e \leq 0) \land (c \leq 0 \lor (c + d = 0 \land e \leq 0))
  \]

determines an invariant of the loop.

For example, substituting the following values in the template leads to invariants:

- \([0, -1, 0]\) \(\rightarrow\) \(n \geq 0\)
- \([1, 0, 0]\) \(\rightarrow\) \(x \geq 0\)
- \([1, -1, 0]\) \(\rightarrow\) \(x - n \leq 0\)
Constraint-based Inv. Inference: Summary

- **Main issues:**
  - choice of invariant templates for which effective decision procedures exist
    - interesting research topic per se, on the brink of undecidability
  - heuristics to extract the “best” invariants from the set of solutions

- **Advantages:**
  - sound & complete (w.r.t. the template)
  - exploit heterogeneous decision procedures together
  - fully automated (possibly except for providing the template)
    - providing the template introduces a “natural” form of user interaction

- **Disadvantages:**
  - suitable mathematical decision theories are usually quite sophisticated
    - hence, hard to extend and customize
  - exact constraint solving is usually quite expensive
  - mostly suitable for algebraic/numeric/scalar invariants
    - requires integration with other techniques to achieve full functional correctness proofs
Dynamic Techniques for Invariant Inference
Dynamic Invariant Inference

- In a nutshell:
  testing of candidate invariants
  - Choose a set of test cases
  - Perform runtime monitoring of candidate invariants
  - If some test run violates a candidate, discard the candidate
  - The surviving candidates are guessed invariant

- Daikon tool, 1999 -- Mike Ernst et al.
- CITADEL: Daikon for Eiffel, 2008 -- Nadia Polikarpova
- AutoInfer for Eiffel (Yi “Jason” Wei et al.)
Dynamic Invariant Inference: Example

```plaintext
dummy_routine (n: NATURAL)
local x: NATURAL do
  from x := 0
  until x ≥ n
  loop x := x + 1 end
end
```

- **Test cases:** \{ n = k | 0 ≤ k ≤ 1000 \}
- **Candidate invariants:**
  - \{ x ≥ c | -1000 ≤ c ≤ 1000 \},
    \{ n ≥ c | -1000 ≤ c ≤ 1000 \}
  - \{ x = c·n + d | -500 ≤ c, d ≤ 500 \}
  - \{ x < n, x ≤ n, x = n, x ≠ n, x ≥ n, x > n \}
  - \{ x ± n ≥ c | -500 ≤ c ≤ 500 \}
  - ...
Dynamic Invariant Inference: Example

\textit{dummy\_routine}(n: \text{NATURAL})
\begin{verbatim}
local x: \text{NATURAL} do
  from x := 0
  until x \geq n
  loop x := x + 1 end
end
\end{verbatim}

- **Survivors (after loop iterations)**:
  - \{ x \geq -c \mid 0 \leq c \leq 1000 \},
    \{ n \geq -c \mid 0 \leq c \leq 1000 \}
  - x \leq n
  - \{ x + n \geq c \mid -500 \leq c \leq 500 \}
  - ...

Dynamic Invariant Inference: Summary

Main issues:
- choose suitable test cases
- handle huge sets of candidate invariants (runtime overhead)
- estimate soundness/quality of survivor predicates
- select heuristically the “best” survivor predicates

Advantages:
- straightforward to implement (at least compared to other techniques)
- guessing is often rather accurate in practice (possibly with some heuristics)
- customizable and rather flexible:
  in principle, whatever you can test you can check for invariance

Disadvantages:
- unsound (educated guessing)
- without heuristics, large amount of useless, redundant predicates
- sensitive to choice of test cases
- some complex candidate invariants are difficult to implement efficiently
Exact Static Techniques for Invariant Inference:

Direct Approach
Direct Static Invariant Inference

- In a nutshell:
  solve the fixpoint equations underlying the program
  
  - $v(i)$: value of variable $v$ at step $i$ of the computation
  - Encode the semantics of loops explicitly and directly as recurrence equations over $v(i)$
  - Solve recurrence equations
  - Eliminate step parameter $i$ to obtain invariant

- 1973 -- Shmuel Katz & Zohar Manna
- 2005 -- Laura Kovacs et al.
Direct Static Invariant Inference: Example

\[
dummy\_\text{routine} (n: \text{NATURAL})
\]
\[
\text{local } x: \text{NATURAL do}
\]
\[
\text{from } x := 0
\]
\[
\text{until } x \geq n
\]
\[
\text{loop } x := x + 1 \text{ end}
\]
\[\text{end}\]

- \(x(i), n(i)\)

- Recurrence relations:

\[
x(i) = \begin{cases} 
0 & i = 0 \\
x(i - 1) + 1 & 0 < i \leq n_0 \\
x(i - 1) & i > n_0 
\end{cases} \quad n(i) = \begin{cases} 
n_0 \geq 0 & i = 0 \\
n(i - 1) & i > 0 
\end{cases}
\]
Direct Static Invariant Inference: Example

\[ x(i) = \begin{cases} 0 & i = 0 \\ x(i - 1) + 1 & 0 < i \leq n_0 \\ x(i - 1) & i > n_0 \end{cases} \]

\[ n(i) = \begin{cases} n_0 \geq 0 & i = 0 \\ n(i - 1) & i > 0 \end{cases} \]

- **Solving recurrence relations:**
  - \( x(i) = \min(n_0, i) \geq 0 \)
  - \( n(i) = n_0 \)
- **Eliminating step parameter \( i \):**
  - \( x(i) - n(i) = \min(n_0, i) - n_0 \leq 0 \), or:
  - \( x - n \leq 0 \), hence:
  - \( 0 \leq x \leq n \)
Direct Static Invariant Inference: Summary

- **Main issues:**
  - in its bare form, more a set of guidelines than a technique
  - step parameter elimination is tricky

- **Advantages:**
  - since semantics is represented explicitly, obtained invariants are often powerful
  - benefits from the programmer's ingenuity
  - additional information about the program can be plugged in

- **Disadvantages:**
  - solving recurrence equations can be very difficult (when possible at all)
  - typically restricted to algebraic/numeric/scalar invariants
APPENDIX – Additional Material
Exact Static Techniques for Invariant Inference:

Approach Based on Abstract Interpretation
Abstract Interpretation for Invariants

- In a nutshell:
  
  \textit{symbolic execution over an abstract domain with guarantee of termination}

- Consider the over-approximation of the value of variables over some coarse abstract domain (instead of their exact values)

- Symbolically execute the program over the abstract domain

- Iterate loops until termination
  
  - termination guaranteed by the nature of the abstract domain or by heuristic cut-offs (widening)

- The final expression is an invariant

- 1976 -- Michael Karr


- ...
Abstract Interpretation for Inv.: Example

dummy_routine (n: NATURAL)
local x: NATURAL do
  from x := 0
  until x ≥ n
  loop x := x + 1 end
end

- Abstract interval domain:
  conjunction of inequalities in the form
  \[ v \leq c \text{ or } c \leq v \]
  for any program variable \( v \) and integer constant \( c \)
- Initially: \( S(0) \triangleq \{ 0 \leq x, x \leq 0, 0 \leq n \} \)
- After one loop iteration: \( S(1) \triangleq \{ 1 \leq x, x \leq 1, 0 \leq n \} \)
- Set of abstract states reached in at most one loop iteration:
  \( S(0) \lor S(1) = \{ 0 \leq x, x \leq 1, 0 \leq n \} \)
Abstract Interpretation for Inv.: Example

\begin{center}
\textbf{dummy\_routine} (n: NATURAL)
\textbf{local} x: NATURAL \textbf{do}
\textbf{from} x := 0 \textbf{until} x \geq n \textbf{loop} x := x + 1 \textbf{end}
\textbf{end}
\end{center}

- Initially: \( S(0) \triangleq \{ 0 \leq x, x \leq 0, 0 \leq n \} \)
- Set of abstract states reached in at most one loop iteration:
  \( S(0) \lor S(1) = \{ 0 \leq x, x \leq 1, 0 \leq n \} \)
- \( S(0) \lor S(1) \) does not subsume \( S(0) \)
  - no fixpoint, keep on iterating
- Abstract states after at most \( k \) loop iterations:
  \( S(0) \lor \ldots \lor S(k) = \{ 0 \leq x, x \leq k, 0 \leq n \} \)
- No convergence as: \( S(0) \lor \ldots \lor S(k) \) does not subsume \( S(0) \lor \ldots \lor S(k-1) \)
Abstract Interpretation for Inv.: Example

dummy_routine (n: NATURAL)
local x: NATURAL do
  from x := 0
  until x ≥ n
  loop x := x + 1 end
end

- Abstract states after at most \( k \) loop iterations:
  \[
  S(0) \lor \ldots \lor S(k) = \{0 \leq x, x \leq k, 0 \leq n\}
  \]

- Apply heuristic over-approximation:
  - relax \( S(0) \lor \ldots \lor S(k) \) by dropping inequalities with growing bounds:
    \[
    S' = \text{widen} \{0 \leq x, x \leq k, 0 \leq n\} = \{0 \leq x, 0 \leq n\}
    \]

- \( S' \) is a fixpoint of the loop iteration

- \( 0 \leq x \land 0 \leq n \) is a loop invariant
  - a very weak one, but more sophisticated choices of abstract domain and/or heuristic over-approximation would yield the “desired” \( 0 \leq x \leq n \)
Abstract Interpretation for Inv.: Summary

- **Main issues:**
  - effective choice of abstract domain
    - trade off: accuracy vs. computational efficiency
  - smart choice of heuristic widening

- **Advantages:**
  - the abstract interpretation framework is quite general and customizable to many different program properties
  - fully automated
  - sound
  - scalable: efficient implementations are possible

- **Disadvantages:**
  - incompleteness, from two sources:
    - invariants can be inexpressible in the abstract domain
    - even if they are expressible, heuristic widening loses completeness
  - requires integration with other techniques to achieve full functional correctness proofs
Statistical Static Techniques for Invariant Inference
The **goal** of the analysis is usually **different** than for other classes of invariant inference techniques:

- inferring likely specific **behavioral specification** (e.g., temporal properties)
- inferring likely violations of **invariance behavioral properties**
- no functional invariants

**Examples of inferred behavioral specs:**

- Location $x$ performs allocation ($A$) of some resource
- Location $y$ performs deallocation ($D$) of some resource
- Every allocated resource is eventually deallocated
Statistical Static Invariant Inference

- In a nutshell: learning of a statistical model
  - **Classify** all possible behaviors (w.r.t. goal properties) of code
  - **Assign prior probabilities** to different behaviors
    - possibly including additional knowledge
  - **Compute cumulative probabilities** of code
    - e.g., through simple symbolic execution
  - **Report likely inferred specifications** or likely invariance violations
    - those with the highest cumulative probabilities

- 2002 -- James Larus et al.
- ...

2002 -- James Larus et al.
...
Statistical Invariant Inference: Example

- **Prior probabilities:**
  - Pr [green] = 0.9 (probability that the sequence is ok)
  - Pr [red] = 0.1 (probability that the sequence gives a bug)

- **Cumulative probabilities**, e.g.:
  - Pr [A, ¬D, D] = \( \prod f(A, \neg D, D) = 0.9 \times \ldots \times \ldots \)
    - if are the various elements of knowledge
Statistical Invariant Inference: Summary

- **Main issues:**
  - choose suitable behavioral properties
  - compute efficiently complex products of probabilities for a huge number of candidate behaviors
  - classify possible behaviors by their probabilities
  - introduce effective *ad hoc* factors

- **Advantages:**
  - reasonably robust w.r.t. the choice of prior probabilities
  - customizable: can incorporate very specific probability factors
  - can handle large “real” programs

- **Disadvantages:**
  - unsound
  - mostly limited to simple behavioral properties
  - best suited for “systems” code
    (where full functional correctness is usually not a concern)