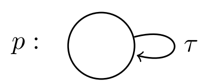
Assignment 11: Bisimulations and Coalgebra

ETH Zurich

1 Bisimulations

1.1 Tasks

1. Consider two CCS processes p and q, depicted as follows:



   (a) Are p and q strongly bisimilar? Justify your answer and provide the strong bisimulation relation, if applicable.

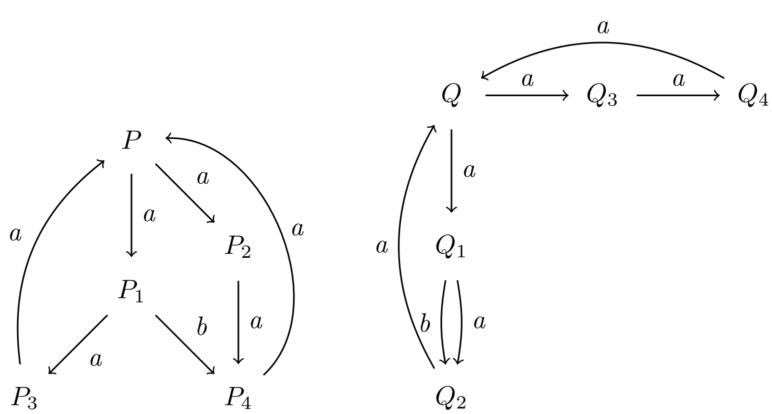
   (b) Are p and q weakly bisimilar? Justify your answer and provide the weak bisimulation relation, if applicable.

   (c) Consider the following CCS processes:

$$\begin{aligned} p' &= p + a \,.\, p_0 \\ p_0 &= b \,.\, p_0 \\ q' &= \overline{a} \,.\, \mathbf{0}, \end{aligned}$$

   where a and $\overline{a}$ are complementary actions, $\mathbf{0}$ stands for the process that cannot perform any action, and p is the process illustrated above.
   Draw the LTSs corresponding to $p' \mid q'$ and $(p' \mid q')\backslash\{a\}$, respectively. Are these LTSs weakly or strongly bisimilar?

2. Consider the following labelled transition system:



   Show that $P \sim Q$ by finding a strong bisimulation $\mathcal{R}$ such that $P \,\mathcal{R}\, Q$.

3. Suppose we have the following definitions of processes

$$\begin{aligned} \mathrm{S} &\stackrel{\text{def}}{=} a.\overline{b}.\mathrm{S} \\ \mathrm{T} &\stackrel{\text{def}}{=} \overline{a}.e.b.\mathrm{T} \\ \mathrm{ST} &\stackrel{\text{def}}{=} (\mathrm{S} \mid \mathrm{T}) \setminus \{a, b\} \end{aligned}$$

Further we have

$$\begin{aligned} \mathrm{U} &\stackrel{\text{def}}{=} e.x.y.\mathrm{U} \\ \mathrm{V} &\stackrel{\text{def}}{=} \overline{x}.\overline{y}.\mathrm{V} \\ \mathrm{UV} &\stackrel{\text{def}}{=} (\mathrm{U} \mid \mathrm{V}) \setminus \{x, y\} \end{aligned}$$

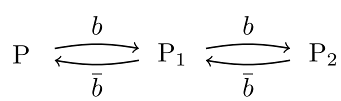
Your task is to

(a) Represent ST and UV as LTSs.

(b) Show that ST and UV are weakly bisimilar.

(c) Suppose we further have $\mathrm{UV}' \stackrel{\text{def}}{=} (\mathrm{U} \mid \mathrm{V}) \setminus \{y\}$. Show that ST and UV′ are not weakly bisimilar.

4. Consider the labeled transition system describing the behavior of a process P:



Furthermore, consider the CCS process Q defined by the following equations:

$$\begin{aligned} \mathrm{Q} &\stackrel{\text{def}}{=} (\mathrm{Q}_1 \mid \mathrm{Q}_2) \setminus \{a\} \\ \mathrm{Q}_1 &\stackrel{\text{def}}{=} a.\bar{b}.\mathrm{Q}_1 \\ \mathrm{Q}_2 &\stackrel{\text{def}}{=} b.\bar{a}.\mathrm{Q}_2 \end{aligned}$$

(a) Draw a labeled transition system that describes the behavior of process Q.

(b) (a) Are the processes P and Q strongly bisimilar?

(b) Are the processes P and Q weakly bisimilar?

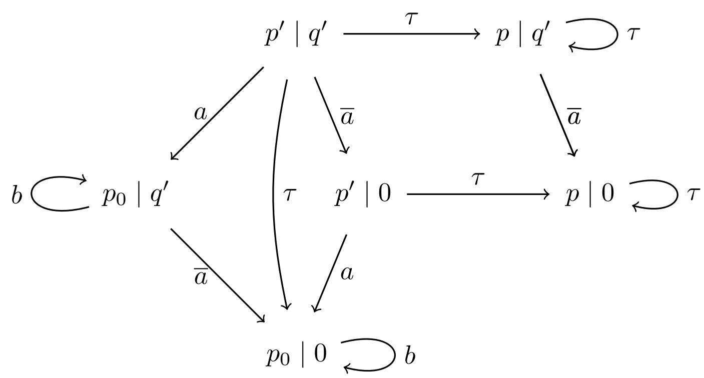
Justify your answers: if yes, give a strong (weak) bisimulation $\mathcal{R}$ such that $\mathrm{P}\,\mathcal{R}\,\mathrm{Q}$; if no, argue why not.

1.2 Solution

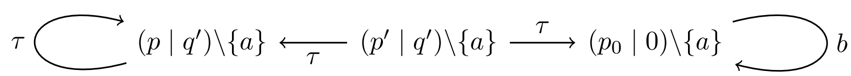
1. The solutions are:
(a) The two processes are not strongly bisimilar, as strong bisimilarity does not abstract from internal behaviour (τ). Here, process p can trigger action τ, whereas q cannot.

(b) Processes p and q are weakly bisimilar, as weak bisimilarity abstracts from internal behaviour. The weak bisimulation relation is $R = \{(p, q)\}$.

(c) The LTS corresponding to $p' \mid q'$ can be depicted as follows:



The LTS corresponding to $(p' \mid q')\backslash\{a\}$ can be depicted as follows:



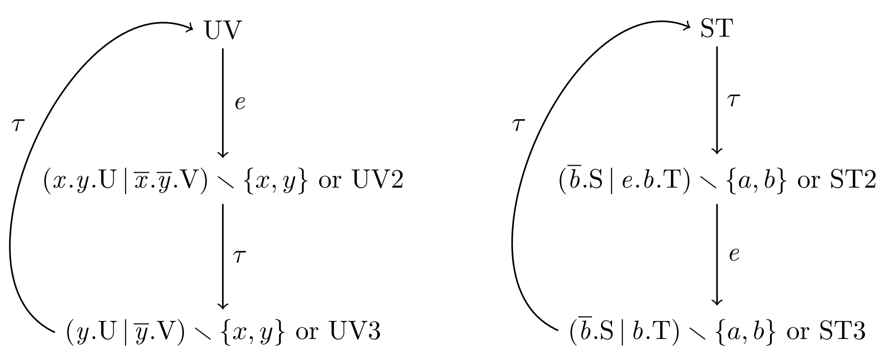
It is easy to see that the LTSs above are neither weakly bisimilar, nor strongly bisimilar, as the second system does not have any transition labelled a or $\overline{a}$.

2. A strong bisimulation $\mathcal{R}$ is given by the following relation:

$$\mathcal{R} = \{(P, Q), (P_1, Q_1), (P_3, Q_2), (P_4, Q_2), (P_2, Q_3), (P_4, Q_4)\}$$

3. The solutions are:

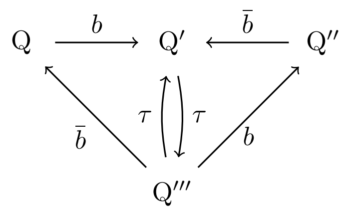
   (a)



   (b) The weak bisimulation here is $\{ST, ST2, ST3\} \times \{UV, UV2, UV3\}$. An alternative weak bisimulation relation is $\{(UV, ST), (UV, ST2), (UV2, ST3), (UV3, ST3)\}$.

   (c) This is no longer a weak bisimulation. Due to the exposure of x, UV' can now make transitions that are impossible in ST.

4. The solutions are:
   1.



$$\text{where}\quad \begin{aligned} \mathrm{Q}' &\stackrel{\text{def}}{=} (\mathrm{Q}_1 \mid \bar{a}.\mathrm{Q}_2) \setminus \{a\} \\ \mathrm{Q}'' &\stackrel{\text{def}}{=} (\bar{b}.\mathrm{Q}_1 \mid \bar{a}.\mathrm{Q}_2) \setminus \{a\} \\ \mathrm{Q}''' &\stackrel{\text{def}}{=} (\bar{b}.\mathrm{Q}_1 \mid \mathrm{Q}_2) \setminus \{a\} \end{aligned}$$

2. (a) The processes P and Q are not strongly bisimilar: if $(P, Q) \in \mathcal{R}$ then must also be $(P_1, Q') \in \mathcal{R}$; however, P_1 has an outgoing b transition, which cannot be matched by Q'.

   (b) The processes P and Q are weakly bisimilar: $\mathcal{R} = \{(P, Q), (P_1, Q'), (P_2, Q''), (P_1, Q''')\}$.

2 Bisimulations up-to

Consider the following non-deterministic automata with state space in **Set** and actions labelled in an alphabet A:

The overlined states $\overline{y}$ and $\overline{v}$ represent accepting states.

2.1 Task

1. Discuss, informally, whether x and u in the above figures are bisimilar. How about their language equivalence?
2. Apply the generalized powerset construction and derive the deterministic automata $(o^\sharp_x, t^\sharp_x)$ and $(o^\sharp_u, t^\sharp_u)$ corresponding to x and u, respectively. Identify a bisimulation relation R stating the language equivalence of x and u.
3. A *bisimulation up-to union* is a relation R on $\mathscr{P}(\mathbf{Set})$ such that whenever $Z\ R\ W$ it holds that:

   $$1.\ o^\sharp(Z) = o^\sharp(W) \quad \text{and} \quad 2.\ \text{for all } a \in A,\ t^\sharp(Z)(a)\ u(R)\ t^\sharp(W)(a).$$

   By $u(R)$ we represent the smallest equivalence relation such that:

   $$\frac{Z\ R\ W}{Z\ u(R)\ W} \qquad \frac{Z_1\ u(R)\ W_1 \quad Z_2\ u(R)\ W_2}{Z_1 \cup Z_2\ u(R)\ W_1 \cup W_2}.$$

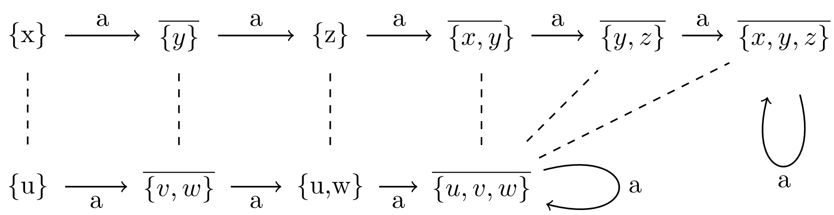
   Moreover, we know that any bisimulation up-to union is contained in a bisimulation. Identify a bisimulation up-to union showing that x and u in the figure are language equivalent. What do you observe?

2.2 Solution

1. x and u are not bisimilar. The reasoning is as follows. Starting from x, after performing an action a leading to an accepting state, it holds that an accepting state can always be

reached by executing an a-sequence of even length: $x \xrightarrow{a} \overline{y} \xrightarrow{a} z \xrightarrow{a} \overline{y} \xrightarrow{a} z \xrightarrow{a} \overline{y} \ldots$. This does not hold for the case of u. Nevertheless, it is easy to see that x and u are language equivalent, as they accept the same a-sequences: a, aaa, $aaaa$, $aaaaa$, $aaaaaa$, $\ldots$.

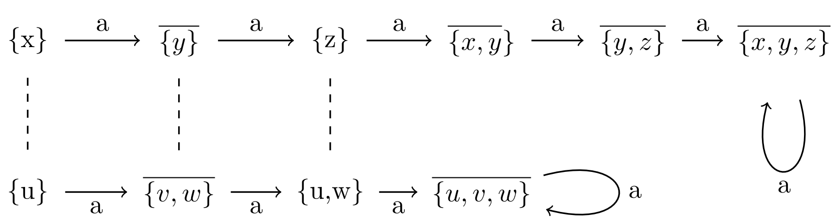
2. The required deterministic automata are illustrated below, together with the bisimulation relation which proves the language equivalence of x and u, as depicted by the dashed lines:



The explicit definitions of $(o^\sharp_x, t^\sharp_x)$ and $(o^\sharp_u, t^\sharp_u)$, respectively, are straightforward. For instance:

$$
\begin{aligned}
o^\sharp_x(\{x\}) &= 0 \\
o^\sharp_x(\{y, z\}) &= 1 \\
t^\sharp_x(\{x\})(a) &= \{y\} \\
t^\sharp_x(\{y, z\})(a) &= \{x, y, z\} \\
o^\sharp_u(\{u, v, w\}) &= 1 \\
t^\sharp_u(\{u, v, w\}) &= \{u, v, w\}
\end{aligned}
$$

3. The bisimulation up-to union proving that x and u are language equivalent is intuitively illustrated by the dashed lines in the figure below:



The equivalence of $\{x, y\}$ and $\{u, v, w\}$ can be immediately deduced from the fact that $\{x\}$ is related to $\{u\}$ and $\{y\}$ to $\{v, w\}$. We observe that the bisimulation up-to union is smaller than the original bisimulation, thus reasoning on language equivalence in terms of bisimulations up-to can be more effective.