Concepts of Concurrent Computation
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Lecture 12: Coalgebra

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Research directions in concurrency

- Rigorous (mathematical) methods
  - for specifying & reasoning on computer systems
  - supported by tool implementations
  - outcome: reliable software/hardware

- Typical scenario:
  - consider:
    - the specification S
    - the implementation I of a system
  - question: does I comply to S?

Theoretical models of computation

(Process) Algebra

Milner: *"Concurrent processes have an algebraic structure"*

P1 **op** P2 ⇒ P1 **op** P2

↑

"constructor"

e.g. CCS: $\boldsymbol{k}$ | . | + | ‖ | \

Structural Operational Semantics (SOS)

e.g. CCS ACT: $\dfrac{}{\alpha.P \xrightarrow{\alpha} P}$

⇒ Behavioral model & equivalence

e.g. LTSs & bisimilarity

Coalgebra

"black-box machine" metaphor

$button : X \rightarrow B(X); \quad X \in \boldsymbol{Set}$

↑

"destructor" (observer)

NO syntax!

B defines both

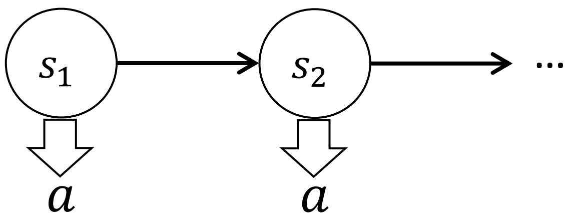
- the behavioral model &
- the equivalence

Systems as coalgebras

$$(X, button : X \rightarrow B(X))$$

Example: Streams

? $(X,\ button : X \to B(X))$

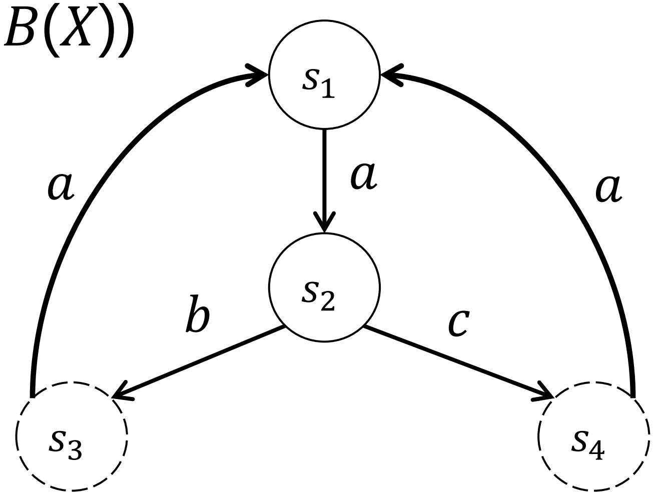


$X = \{s_i \mid i \geq 1\}$
$button = < hd, tl >$
$hd(s_i) = a$ $\quad hd: X \to A$ $\quad A = \{a\}$
$tl(s_i) = s_{i+1}$ $\quad tl: X \to X$

$(X, < hd, tl >: X \to A \times X)$
$B(X) = A \times X$

Example: Deterministic Automata (DAs)

? $(X, button : X \to B(X))$



$X = \{s_1, s_2, s_3, s_4\}$
$button = < o, t >$

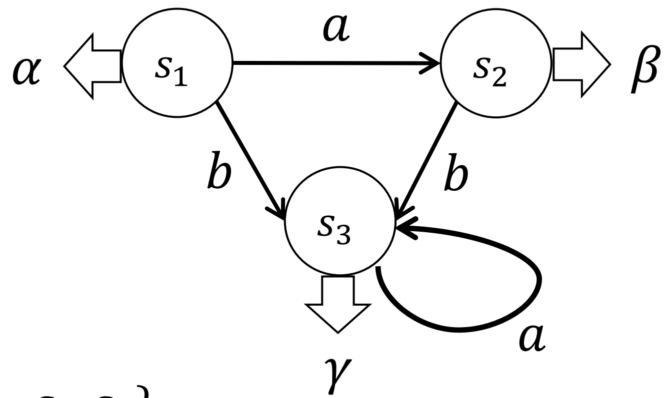
$o: X \to 2 \qquad 2 = \{0,1\}$
$t: X \to X^A \quad X^A = \{f: A \to X\} \quad A = \{a, b, c\}$

$(X, < o, t >: X \to 2 \times X^A)$
$B(X) = 2 \times X^A$

$o(s_1) = 0 \qquad o(s_2) = 0 \qquad o(s_3) = 1 \quad o(s_4) = 1$
$t(s_1)(a) = s_2$
$t(s_2)(b) = s_3 \quad t(s_2)(c) = s_4$
$t(s_3)(a) = s_1 \quad t(s_4)(a) = s_1$

Example: Moore machines

? $(X,\ button : X \to B(X))$



$X = \{s_1, s_2, s_3\}$

$button = < o, t >$

$o(s_1) = \alpha \qquad o(s_2) = \beta \qquad o(s_3) = \gamma$

$t(s_1)(a) = s_2 \quad t(s_1)(b) = s_3$

$t(s_2)(b) = s_3$

$t(s_3)(a) = s_3$

$o: X \to O \quad O = \{\alpha, \beta, \gamma\}$

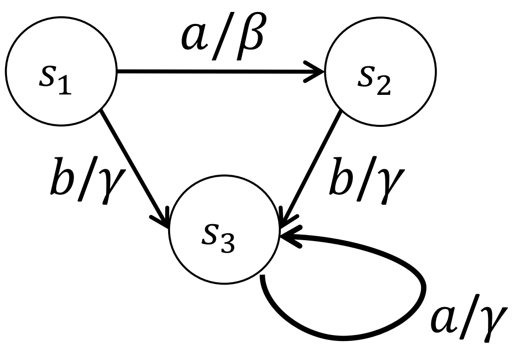
$t: X \to X^A \quad A = \{a, b\}$

$(X, < o, t >: X \to O \times X^A)$

$B(X) = O \times X^A$

Example: Mealy machines

? $(X, \mathit{button} : X \rightarrow B(X))$



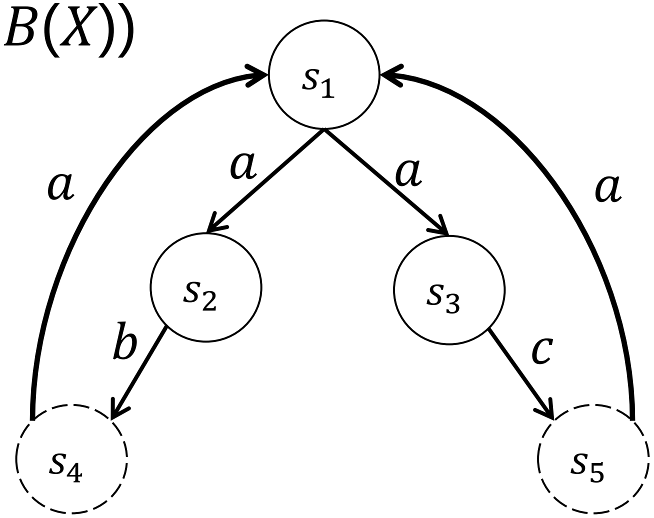
$\mathit{button}(s_1)(a) = < \beta, s_2 >$ $\mathit{button}(s_1)(b) = < \gamma, s_3 >$
$\mathit{button}(s_2)(b) = < \gamma, s_3 >$
$\mathit{button}(s_3)(a) = < \gamma, s_3 >$

$(X, \delta: X \rightarrow (O \times X)^A)$
$B(X) = (O \times X)^A$

$X = \{s_{1,} s_{2,} s_3\}$
$A = \{a, b\}$ $O = \{\beta, \gamma\}$
$\mathit{button} = \delta : X \rightarrow (O \times X)^A$

Example: Nondeterministic Automata (NAs)

? $(X, button : X \rightarrow B(X))$



$X = \{s_1, s_2, s_3, s_4, s_5\}$

$button = < o, t >$

$o(s_1) = 0 \qquad o(s_4) = 1 \quad \ldots$

$t(s_1)(a) = \{s_2, s_3\}$

$t(s_2)(b) = \{s_4\}$

$t(s_4)(a) = \{s_1\}$

...

$o: X \rightarrow 2 \qquad 2 = \{0,1\}$

$t: X \rightarrow (Pow\ X)^A \qquad A = \{a, b, c\}$

$(X, < o, t >: X \rightarrow 2 \times (Pow\ X)^A)$

$B(X) = 2 \times (Pow\ X)^A$

$Pow\ -\ powerset\ functor$

$e.g.\ Pow\ \{1, 2\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Uniform representation of systems

$$B(X) ::= O \mid X \mid B(X)^A \mid Pow\, B(X) \mid B(X) \times B(X) \mid B(X) + B(X)$$

$$(X, button : X \rightarrow B(X))$$

$$Streams\colon\ (X, < hd, tl >: X \rightarrow A \times X)$$

$$DAs\colon\ (X, < o, t >: X \rightarrow 2 \times X^A)$$

$$Moore\ machines\colon\ (X, < o, t >: X \rightarrow O \times X^A)$$

$$Mealy\ machines\colon\ (X, \delta: X \rightarrow (O \times X)^A)$$

$$(finite)\ NAs\colon\ (X, < o, t >: X \rightarrow 2 \times \left(Pow_{fin}\, X\right)^A)$$

$$(finite) LTSs\colon\ (X, t: X \rightarrow \left(Pow_{fin} X\right)^A)$$

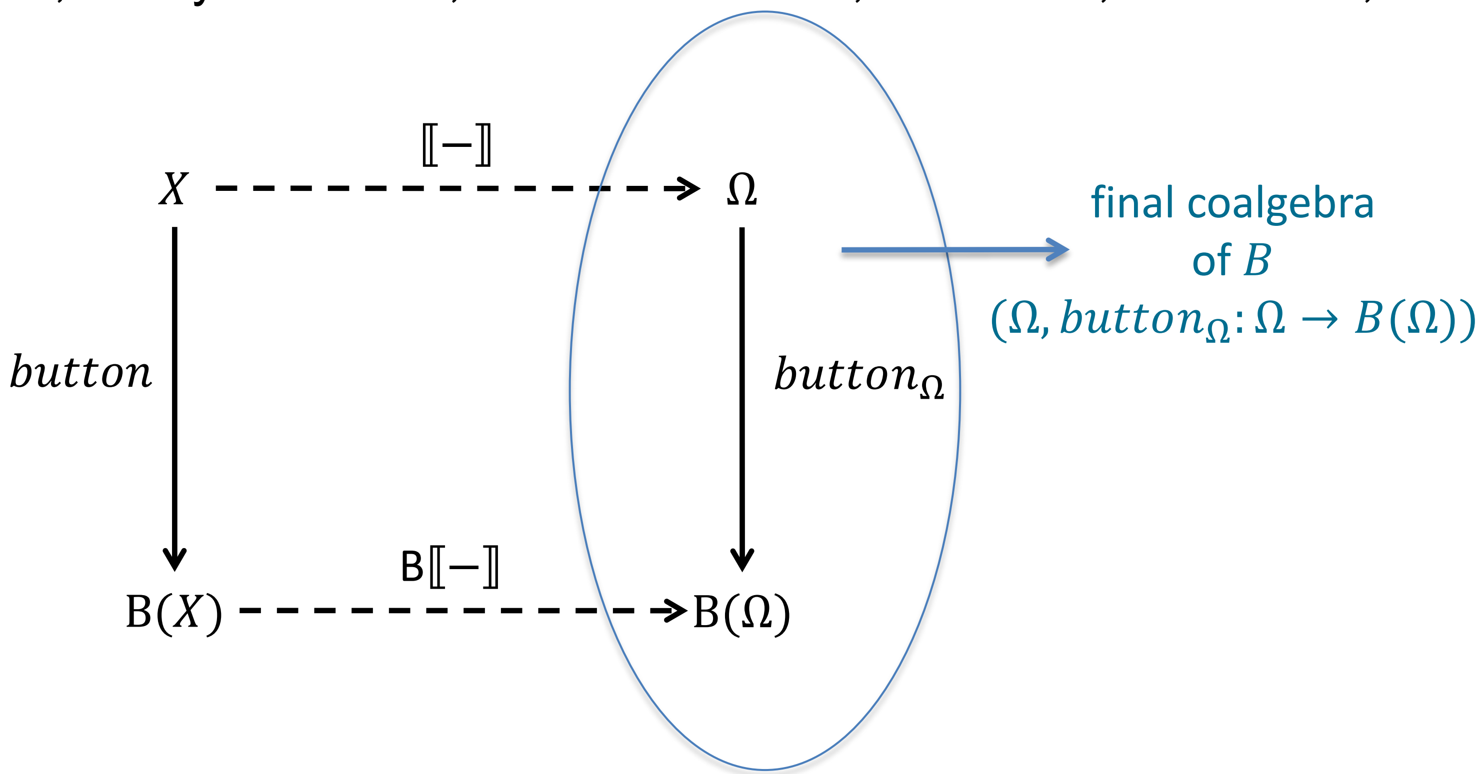
$$divergent\ LTSs\colon\ (X, t: X \rightarrow (1 + Pow\ X)^A)$$

...

Behaviors of systems as coalgebras

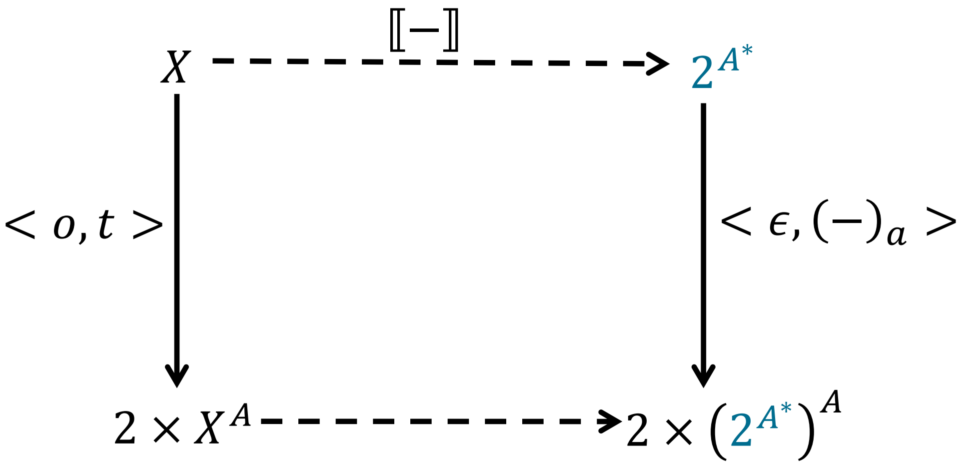
Final coalgebras

- Behaviors of systems “captured” by final coalgebras
  - e.g. DAs, Mealy machines, Moore machines, finite NAs, finite LTSs, …



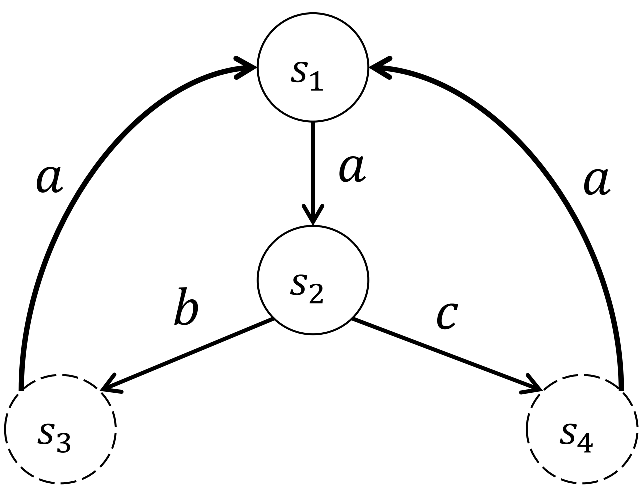
- $[\![-]\!]$ is the (unique) coalgebra homomorphism
  - making the above diagram commute (diagram chasing)
  - i.e. $button_\Omega \circ [\![-]\!] = B[\![-]\!] \circ button$
- Intuition: $[\![x]\!]$ maps x in X to the behavior of x in Ω

Example: DAs $(X, < o, t >: X \to 2 \times X^A)$



- 2^{A^*} - the set of languages over A

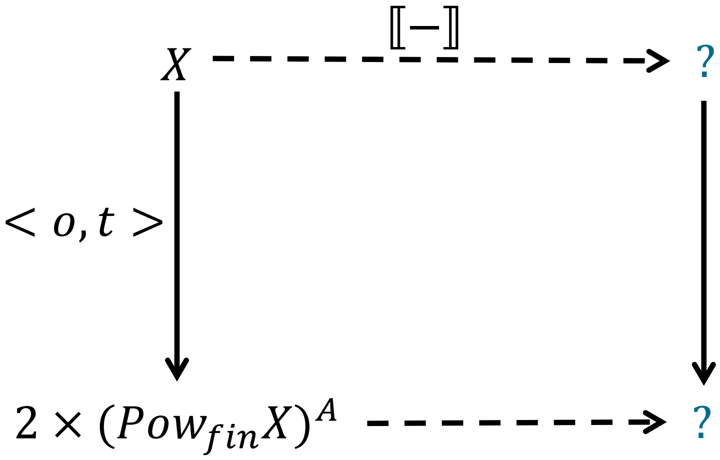
$$\epsilon(L) = 1 \,?\, 0 : \varepsilon \in L$$
$$(L)_a = \{w \in A^* \mid aw \in L\}$$
$$[\![x]\!](\varepsilon) = o(x)$$
$$[\![x]\!](aw) = [\![t(x)(a)]\!](w)$$

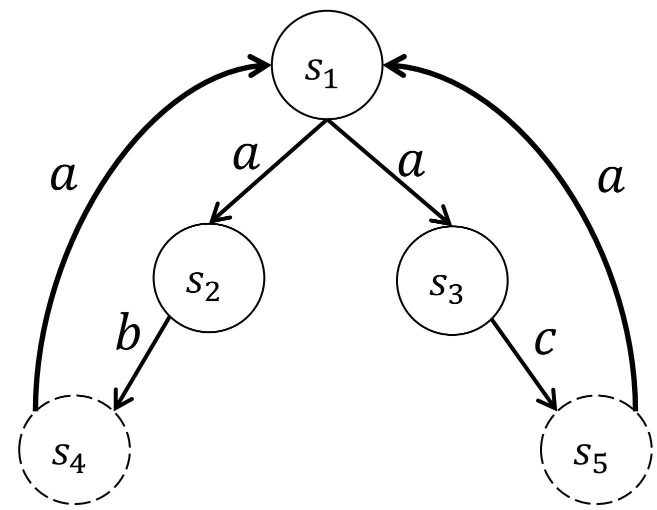


$$s_1 \overset{[\![-]\!]}{\dashrightarrow} \{ab, ac, abaab, acaab, \dots\}$$
$$s_3 \overset{[\![-]\!]}{\dashrightarrow} \{\varepsilon, aab, aac, aacaab, \dots\}$$

Example: finite NAs

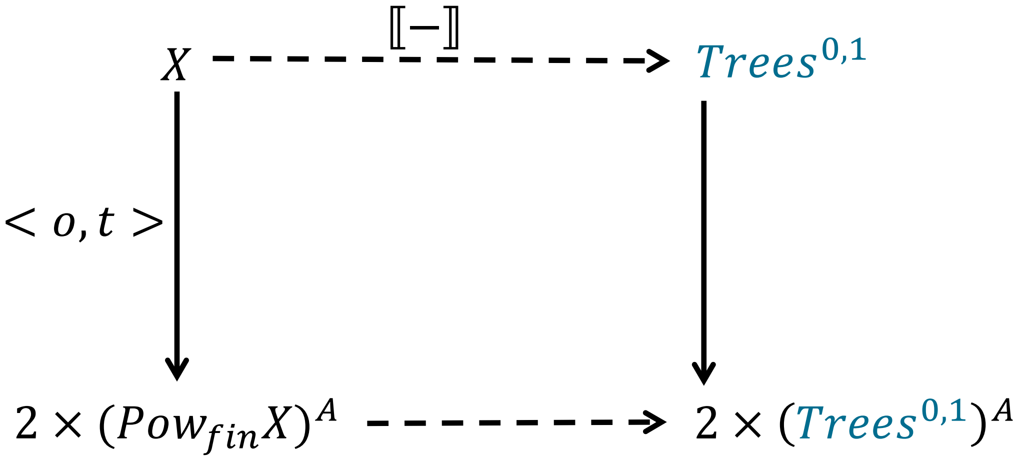
$$(X, < o, t >: X \to 2 \times \left(Pow_{fin}X\right)^A)$$

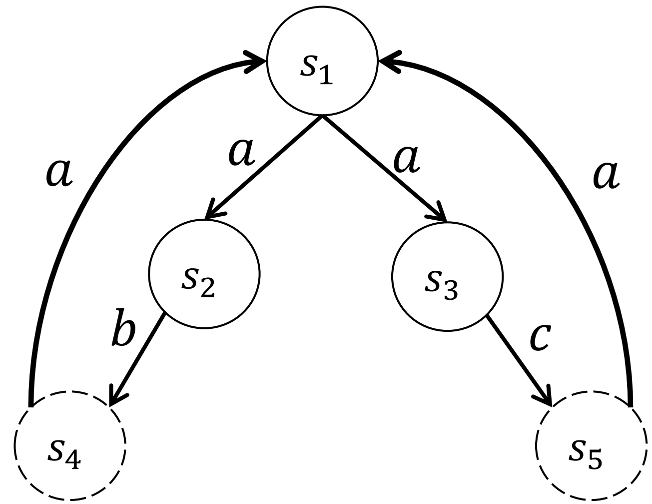


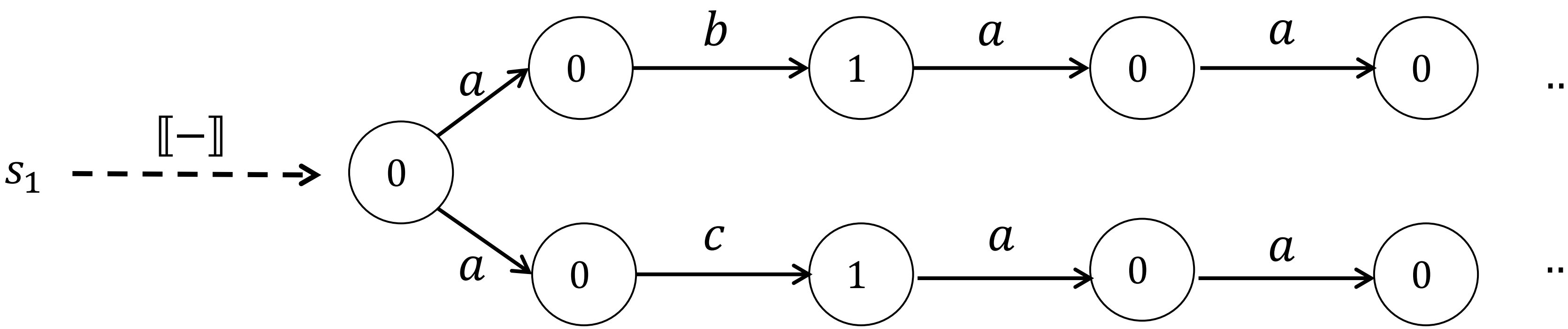


Example: finite NAs

$$(X, < o, t >: X \to 2 \times \left(Pow_{fin}X\right)^A)$$



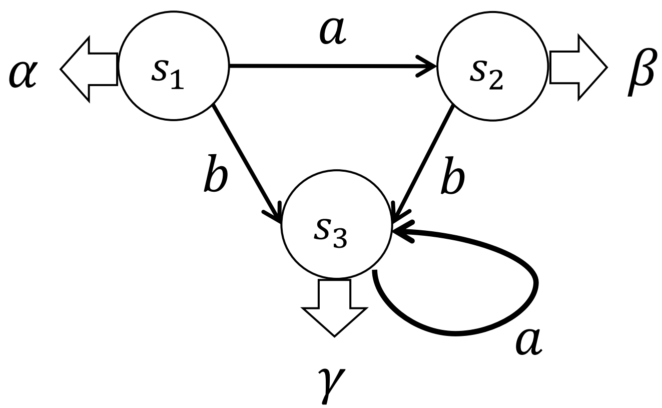




Example: Moore & Mealy machines

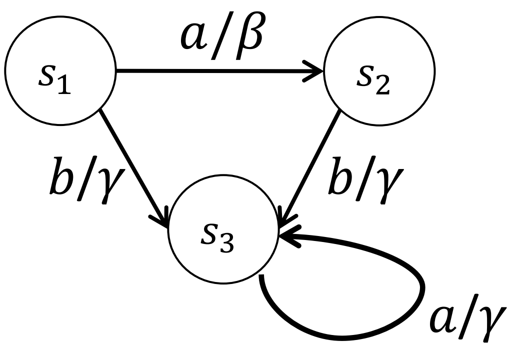
Moore

$(\boldsymbol{X}, <\boldsymbol{o}, \boldsymbol{t}>: \boldsymbol{X} \to \boldsymbol{O} \times \boldsymbol{X}^{\boldsymbol{A}})$

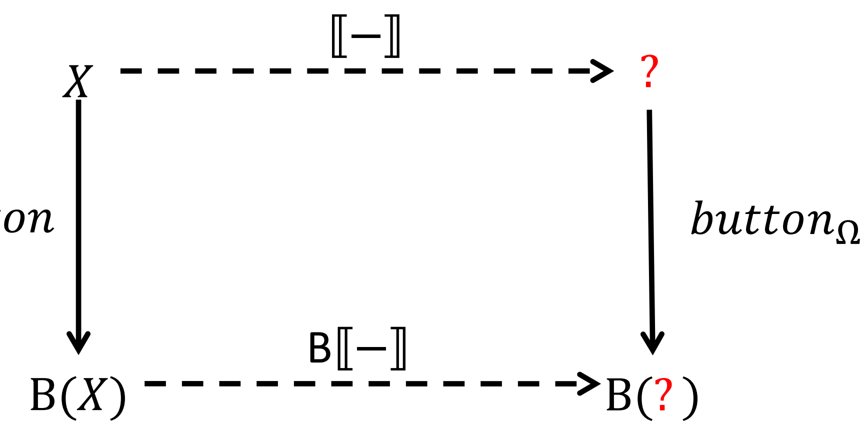


Mealy

$(\boldsymbol{X}, \boldsymbol{\delta}: \boldsymbol{X} \to (\boldsymbol{O} \times \boldsymbol{X})^{\boldsymbol{A}})$



O^{A^*}



O^{A^+}

Uniform representation of behaviors

Final coalgebras: $(\Omega, button_\Omega : \Omega \rightarrow B(\Omega))$

$$B(X) ::= O \mid X \mid B(X)^A \mid Pow_{fin}\, B(X) \mid B(X) \times B(X) \mid B(X) + B(X)$$

DAs

$$(X, < o, t >: X \rightarrow 2 \times X^A) \xrightarrow{[\![-]\!]} (2^{A^*}, < \epsilon, (-)_a >: 2^{A^*} \rightarrow 2 \times \left(2^{A^*}\right)^A)$$

$finite\ NAs$

$$(X, < o, t >: X \rightarrow 2 \times \left(Pow_{fin} X\right)^A) \xrightarrow{[\![-]\!]} (Trees^{0,1}, < o_\Omega, t_\Omega >: Trees^{0,1} \rightarrow 2 \times \left(Pow_{fin} Trees^{0,1}\right)^A)$$

$Moore$

$$(X, < o, t >: X \rightarrow O \times X^A) \xrightarrow{[\![-]\!]} (O^{A^*}, < o_\Omega, t_\Omega >: O^{A^*} \rightarrow O \times (O^{A^*})^A)$$

$Mealy$

$$(X, \delta: X \rightarrow (O \times X)^A) \xrightarrow{[\![-]\!]} (O^{A^+}, \delta_\Omega: O^{A^+} \rightarrow (O \times O^{A^+})^A)$$

…

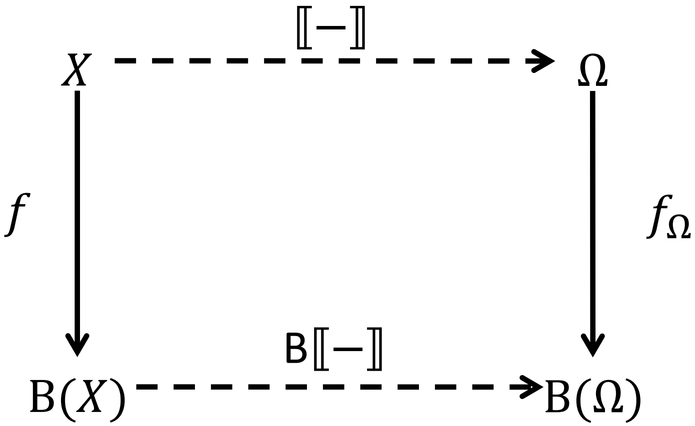
Uniformity

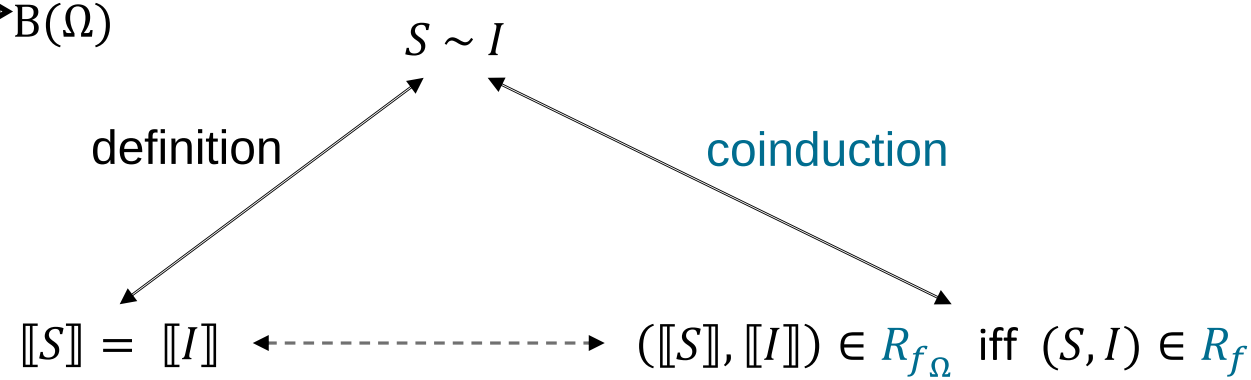
- Each behavioral functor B induces:
  - a type of system
    - e.g. DAs, finite NAs, Moore …
  - the behavior of a certain system type
    - e.g. 2^{A^*}, $Trees^{0,1}$, O^{A^*} ...
  - a corresponding notion of behavioral equivalence
    - e.g. DAs & language equivalence, finite NAs & tree equivalence, Moore machines & “decorated” language equivalence

Behavioral equivalence & Coinduction

Coinduction proof principle

- Idea: reason on behavioral equivalence (~) by
  - bisimulation construction (algorithmic, thus suitable for automation!)





Intuition: R_{f_Ω} and R_f are bisimulations if they are "closed" w.r.t. f_Ω and f, respectively.

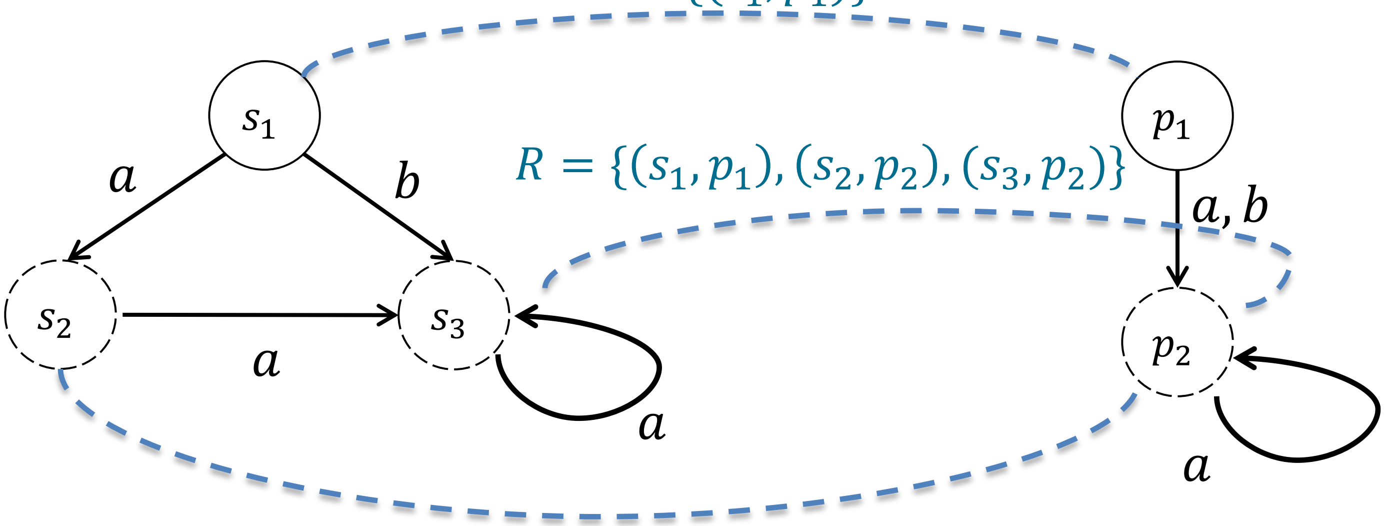
Example: DAs

$o(s_1) = o(p_1) = 0$

$t(s_1)(a) = s_2 \quad t(p_1)(a) = p_2$

$t(s_1)(b) = s_3 \quad t(p_1)(b) = p_2$

$R = \{(s_1, p_1)\}$



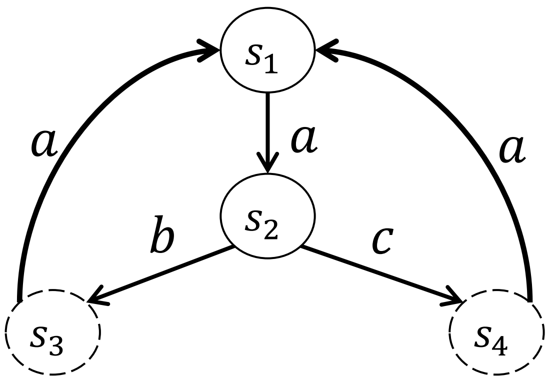
$R = \{(s_1, p_1), (s_2, p_2)\}$

$? \, s_1 \sim p_1$ YES!

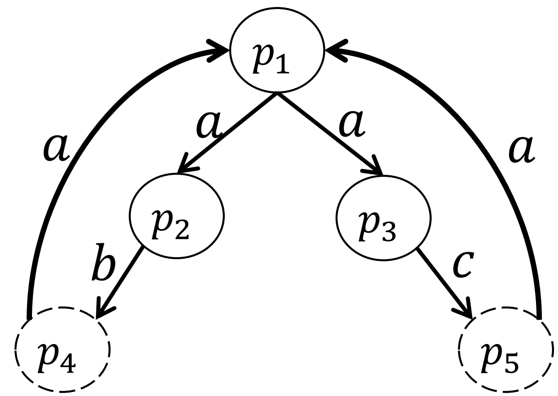
$$\xLeftrightarrow{coinduction} (s_1, p_1) \in R \xLeftrightarrow{def. \sim} [\![s_1]\!] = [\![p_1]\!] \Longleftrightarrow$$

$$L(s_1) = L(p_1) = a^+ | b a^*$$

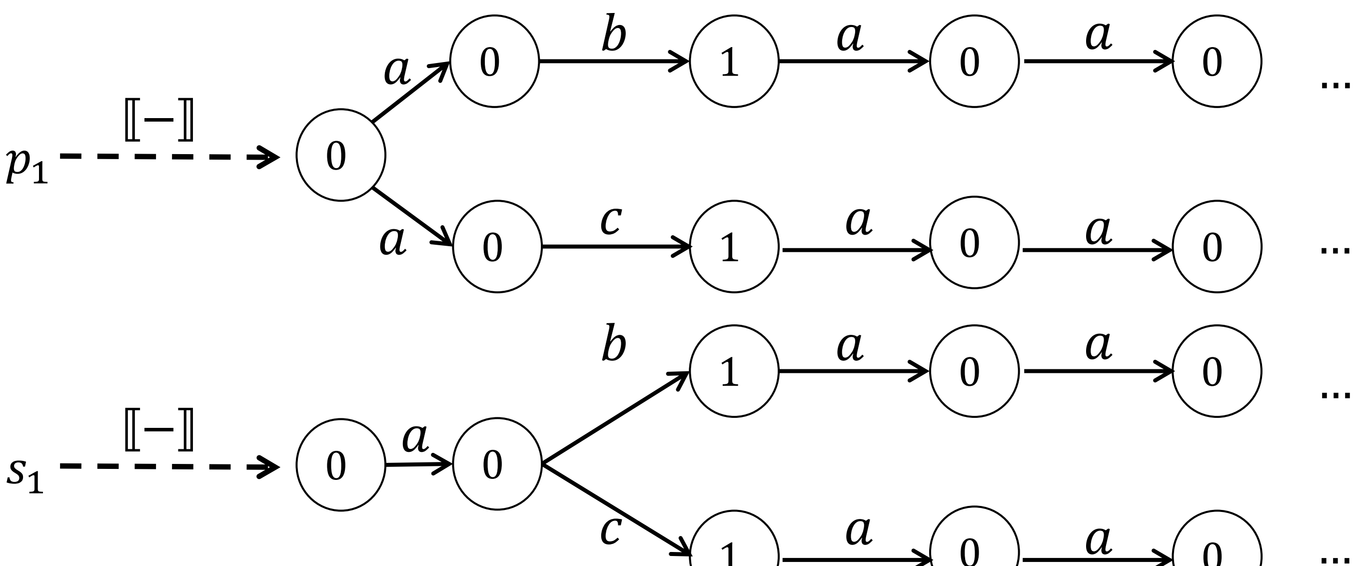
Example: finite NAs



? $s_1 \sim p_1$

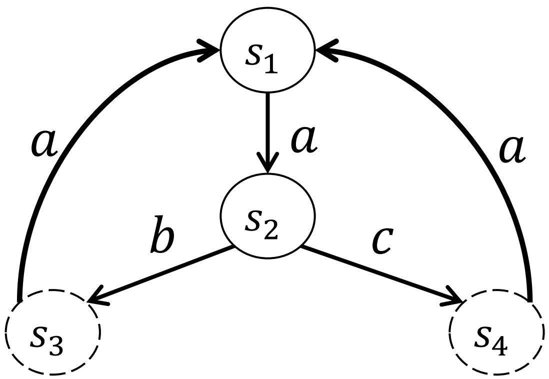


$(\neg \exists R \; a \; bisimulation \; relation). (s_1, p_1) \in R$ hence, s_1 and p_1 are not beh. equiv.

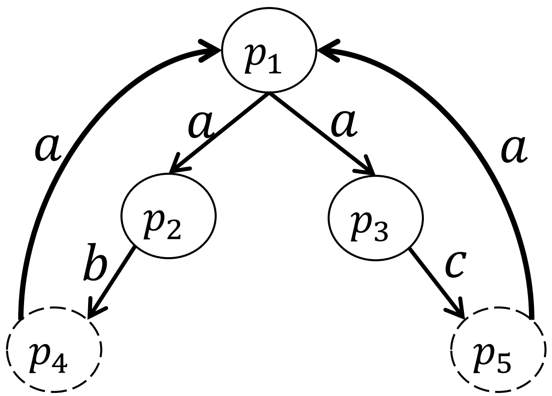


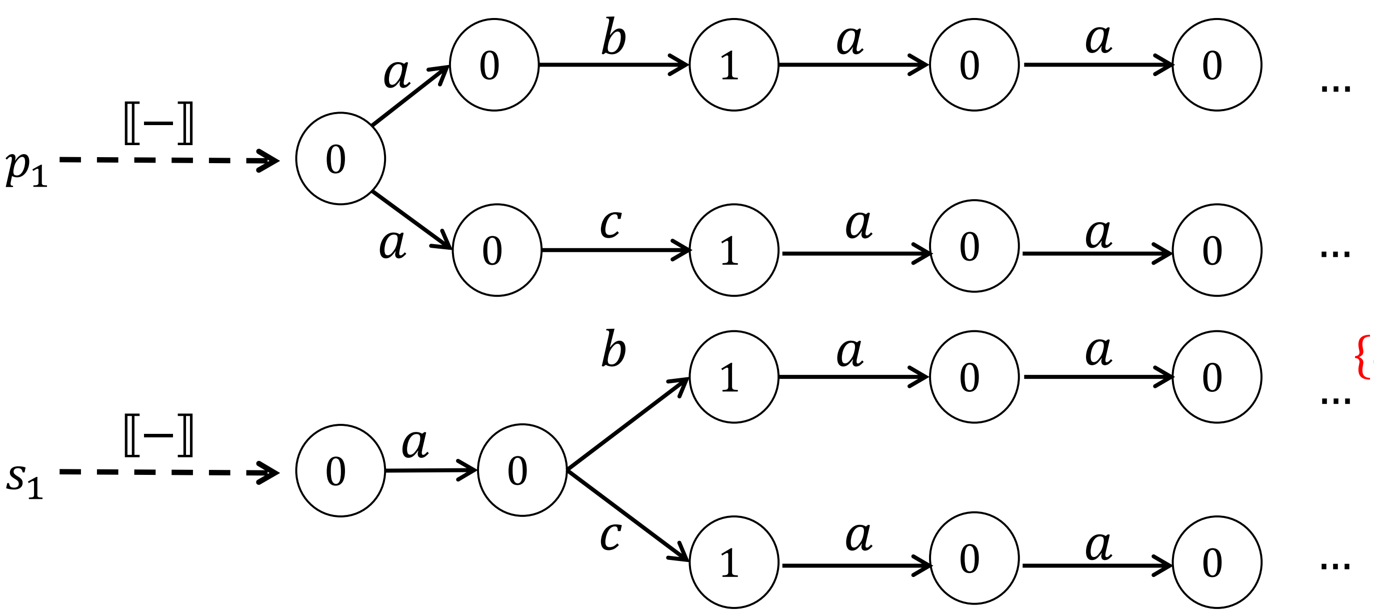
Coalgebraic
Powerset Construction (PC)

Bisimilarity sometimes too fine



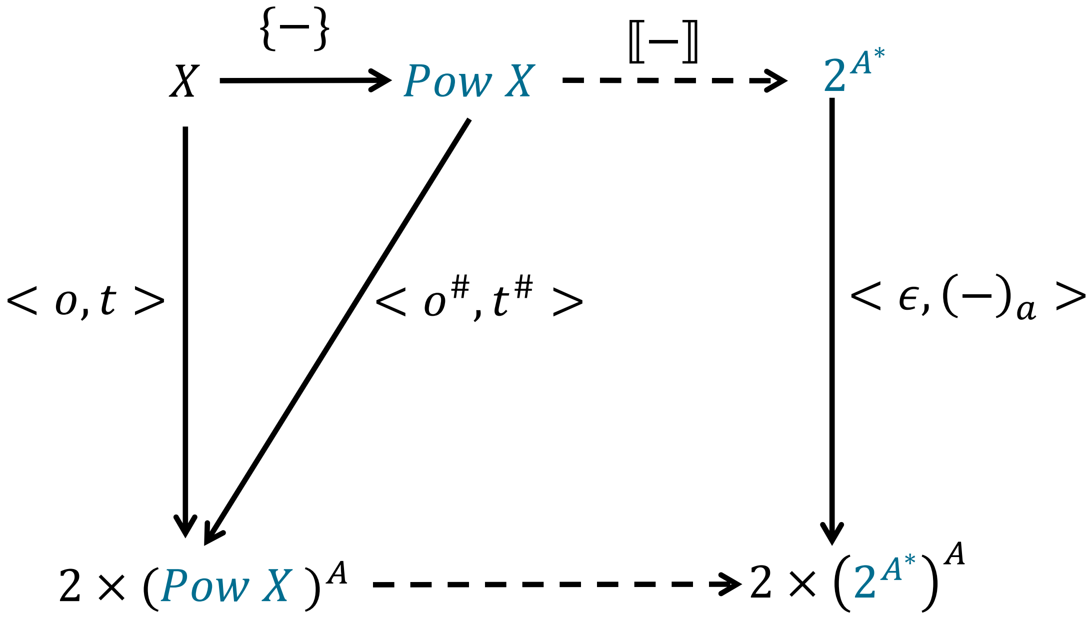
$\neg\, s_1 \sim p_1$





$L(p_1) =$
$L(s_1) =$
$\{ab, ac, abaab, abaac, \dots\}$
!!!

NAs & PC

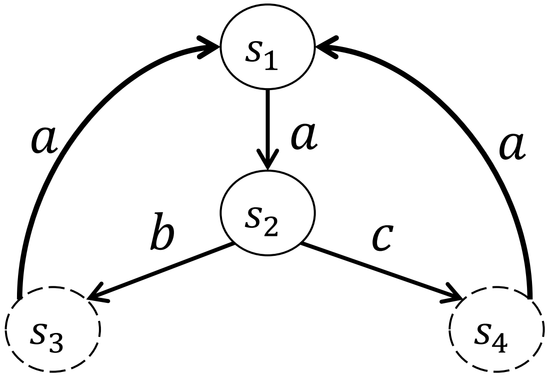


$$x \xrightarrow{behavior} [\![\{x\}]\!] = L(x) \cong 2^{A^*}$$

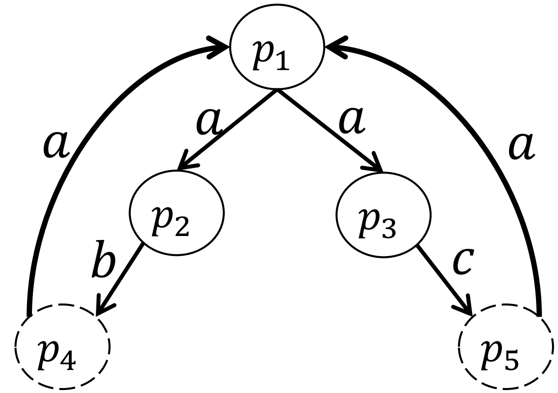
$$o^{\#}(Y) = \bigvee_{y \in Y} o(y) \qquad [\![Y]\!](\varepsilon) = o^{\#}(Y)$$

$$t^{\#}(Y)(a) = \bigcup_{y \in Y} t(y)(a) \qquad [\![Y]\!](aw) = [\![t^{\#}(Y)(a)]\!](w)$$

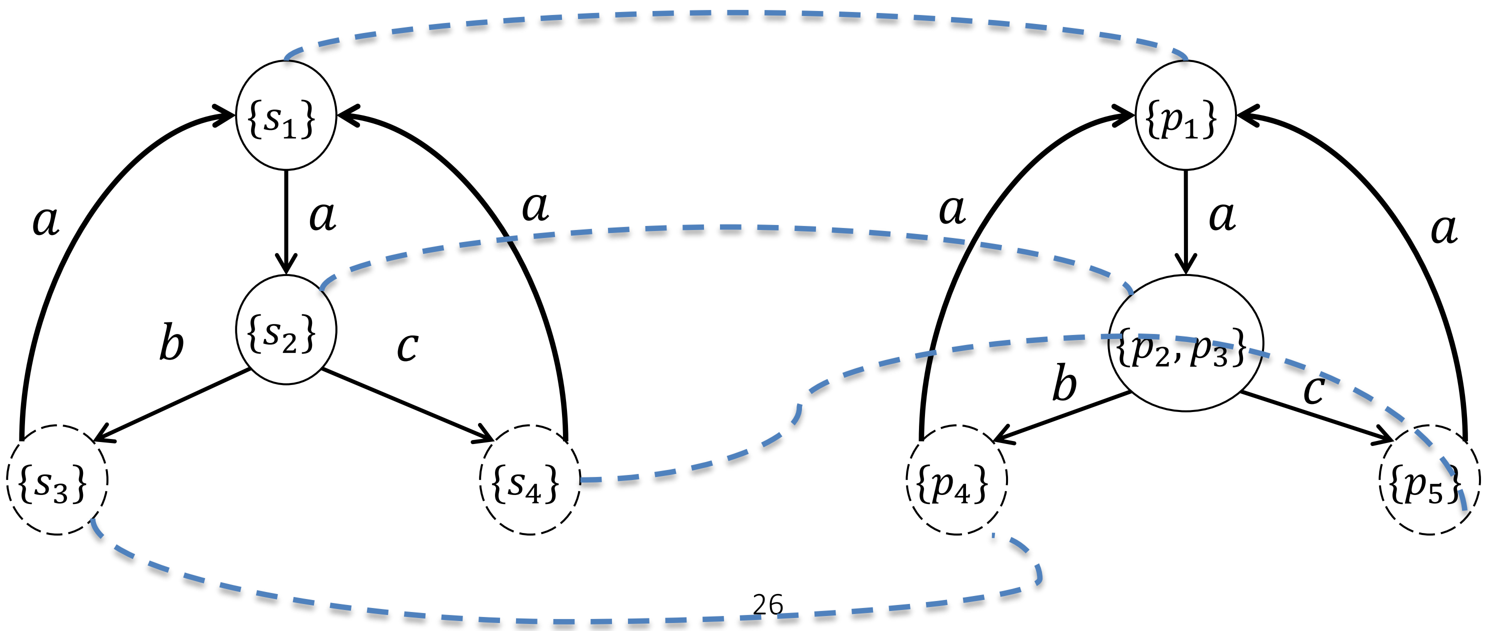
NAs & PC - example



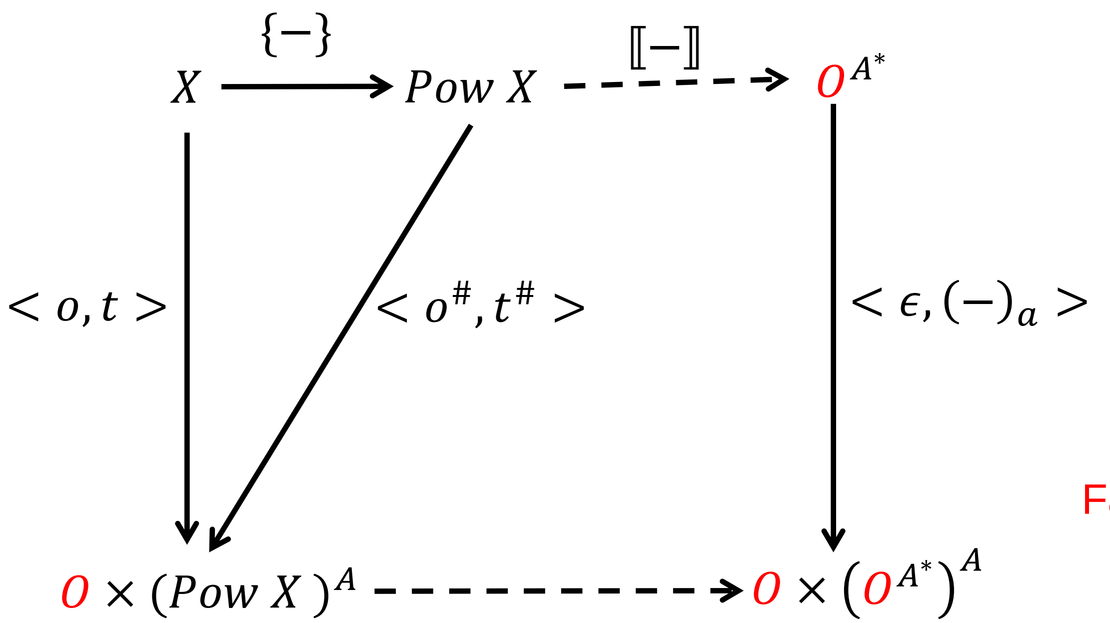
$L(s_1) = L(p_1)$



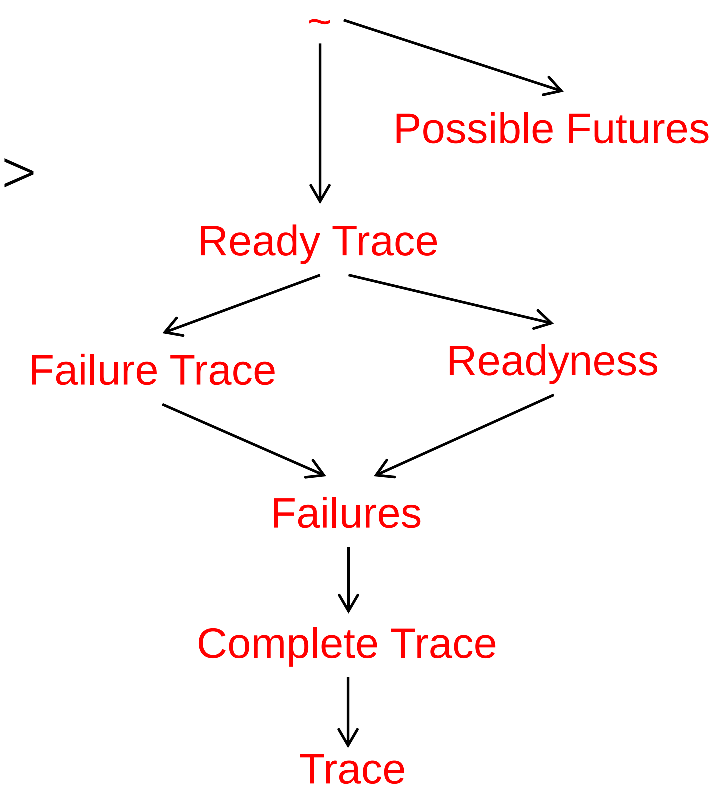
$$L(s_1) = L(p_1) \overset{PC}{\Longleftrightarrow} [\![\{s_1\}]\!] = [\![\{p_1\}]\!] \overset{coinduction}{\Longleftrightarrow} (\{s_1\},\{p_1\}) \in R$$



Generalized PC



van Glabbeek spectrum



$$o^{\#}(Y) = \bigvee_{y \in Y} o(y) \qquad [\![Y]\!](\varepsilon) = o^{\#}(Y)$$

$$t^{\#}(Y)(a) = \bigcup_{y \in Y} t(y)(a) \qquad [\![Y]\!](aw) = \left[\!\left[t^{\#}(Y)(a)\right]\!\right](w)$$

Coalgebra

- Systems as coalgebras
  - uniform representation in terms of a functor B
- System behaviors as final coalgebras
  - uniformly induced by the behavior functor B
- The coinduction proof principle
  - uniform reasoning on behavioral equivalence by bisimulation construction (suitable for automation)
- The (generalized) powerset construction
  - shifting from bisimilarity to language equivalence