Software Verification

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Lecture 2: Axiomatic semantics

Program Verification: the very idea

P: a program

```
max (a, b: INTEGER): INTEGER
    do
        if a > b then
            Result := a
        else
            Result := b
        end
    end
```

S: a specification

```
require
    true
ensure
    Result >= a
    Result >= b
```

Does $P \models S$ hold?

The Program Verification problem:

- Given: a program P and a specification S
- Determine: if every execution of P, for every value of input parameters, satisfies S

What is a theory?

(Think of any mathematical example, e.g. elementary arithmetic)

A theory is a mathematical framework for proving properties about a certain object domain

Such properties are called theorems

Components of a theory:

- Grammar (e.g. BNF), defines well-formed formulae (WFF)
- Axioms: formulae asserted to be theorems
- Inference rules: ways to derive new theorems from previously obtained theorems, which can be applied mechanically

Soundness and completeness

How do we know that an axiomatic semantics (or *logic*) is “right”?

- Sound: every theorem (i.e., deduced property) is a true formula
- Complete: every true formula can be established as a theorem (i.e., by applying the inference rules).
- Decidable: there exists an effective (terminating) process to establish whether an arbitrary formula is a theorem.

Notation

Let f be a well-formed formula

Then

$$\vdash f$$

expresses that f is a theorem

Inference rule

An inference rule is written

$$\frac{f_1, \quad f_2, \ldots, \; f_n}{f_0}$$

It expresses that if f_1, f_2, … f_n are theorems, we may infer f_0 as another theorem

Example inference rule

"Modus Ponens" (common to many theories):

$$\frac{p, \quad p \Rightarrow q}{q}$$

How to obtain theorems

Theorems are obtained from the axioms by zero or more* applications of the inference rules.

*Finite of course

Example: a simple theory of integers

Grammar: Well-Formed Formulae are boolean expressions

- $i1 = i2$
- $i1 < i2$
- $\neg\ b1$
- $b1 \Rightarrow b2$

where b1 and b2 are boolean expressions, i1 and i2 integer expressions

An integer expression is one of

- 0
- A variable n
- f’ where f is an integer expression
  (represents “successor”)

An axiom and axiom schema

$\vdash 0 < 0'$

$\vdash f < g \Rightarrow f' < g'$

An inference rule

$$\frac{P\ (0), \quad P\ (f) \Rightarrow P\ (f')}{P\ (f)}$$

Axiomatic semantics

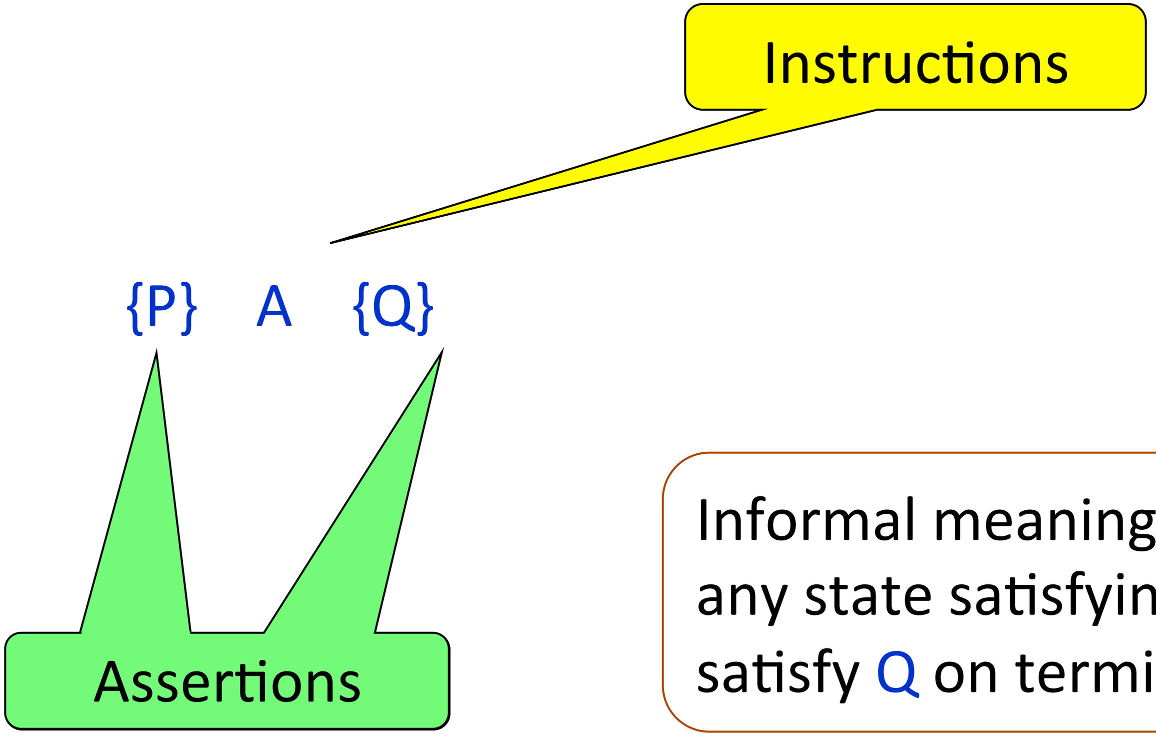
Floyd (1967), Hoare (1969), Dijkstra (1978)

Purpose:

- Describe the effect of programs through a theory of the underlying programming language, allowing proofs

The theories of interest

Grammar: a well-formed formula is a "Hoare triple"



Informal meaning: A, started in any state satisfying P, will satisfy Q on termination

Software correctness (a quiz)

Consider

$$\{P\} \quad A \quad \{Q\}$$

Take this as a job ad in the classifieds

Should a lazy employment candidate hope for a weak or strong P ?
What about Q ?

Two "special offers":

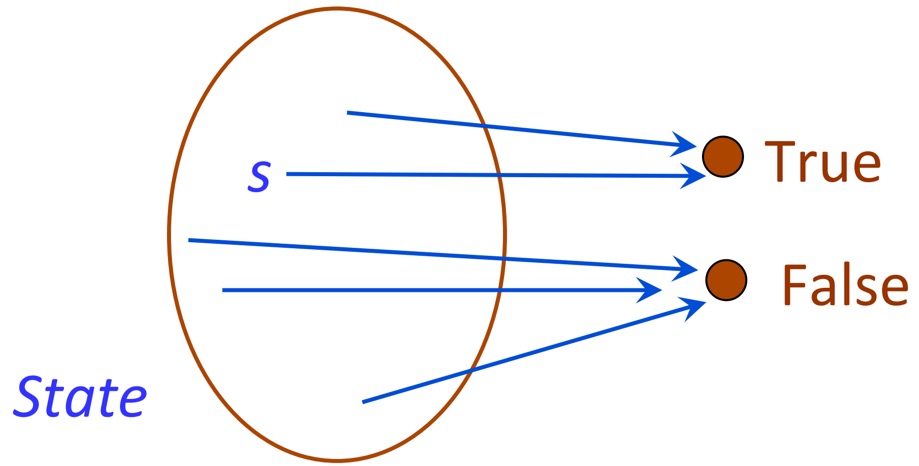
- 1. $\{False\}$ $\quad A \quad$ $\{...\}$
- 2. $\{...\}$ $\quad A \quad$ $\{True\}$

Axiomatic semantics

"Hoare semantics" or "Hoare logic": a theory describing the partial correctness of programs, plus termination rules

What is an assertion?

Predicate (boolean-valued function) on the set of computation states



True: Function that yields True for all states

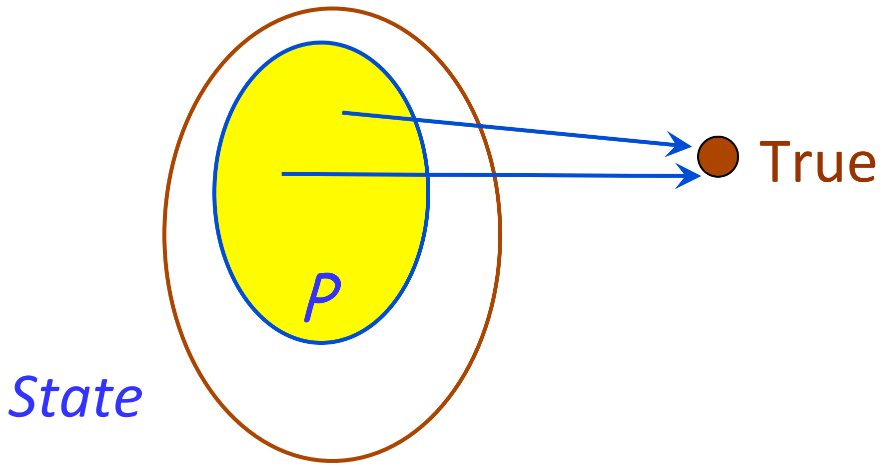
False: Function that yields False for all states

P **implies** Q: means $\forall\ s : State,\ P(s) \Rightarrow Q(s)$

and so on for other boolean operators

Another view of assertions

We may equivalently view an assertion P as a subset of the set of states (the subset where the assertion yields True):



True: Full *State* set

False: Empty subset

implies: subset (inclusion) relation

and: intersection **or**: union

Application to a programming language: Eiffel

```
extend (new : G ; key : H )
        -- Assuming there is no item of key key,
        -- insert new with key ; set inserted.
    require
        key_not_present: not has (key)
    ensure
        insertion_done: item (key) = new
        key_present: has (key)
        inserted: inserted
        one_more: count = old count + 1
```

The case of postconditions

Postconditions are often predicates on two states

Example (Eiffel, in a class *COUNTER*):

```
increment
    require
        count >= 0
    ...
    ensure
        count = old count + 1
```

Partial vs total correctness

$$\{P\} \quad A \quad \{Q\}$$

Total correctness:

- A, started in any state satisfying P, will terminate in a state satisfying Q

Partial correctness:

- A, started in any state satisfying P, will, *if it terminates*, yield a state satisfying Q

Elementary mathematics

Assume we want to prove, on integers

$$\{x > 0\}\ A\ \{y \geq 0\} \qquad [1]$$

but have actually proved

$$\{x > 0\}\ A\ \{y = z \wedge 2\} \qquad [2]$$

We need properties from other theories, e.g. arithmetic

"EM": Elementary Mathematics

The mark [EM] will denote results from other theories, taken (in this discussion) without proof

Example:

$y = z$ ^ 2 **implies** $y \geq 0$ [EM]

Rule of consequence

$$\frac{\{P\}\ A\ \{Q\}, \quad P' \textbf{ implies } P, \quad Q \textbf{ implies } Q'}{\{P'\}\ \ A\ \ \{Q'\}}$$

Example: $\{x > 0\}\ y := x + 2\ \{y > 0\}$

Rule of conjunction

$$\frac{\{P\}\ A\ \{Q\},\quad \{P\}\ A\ \{R\}}{\{P\}\quad A\quad \{Q\ \textbf{and}\ R\}}$$

Example: {True} x := 3 {x > 1 and x > 2}

Axiomatic semantics for a programming language

Example language: Graal (from *Introduction to the theory of Programming Languages*)

Scheme: give an axiom or inference rule for every language construct

Skip

{P} **skip** {P}

Abort

{False} abort {P}

Sequential composition

$$\frac{\{P\}\ A\ \{Q\}, \quad \{Q\}\ B\ \{R\}}{\{P\}\ A\ ;\ B\ \{R\}}$$

Example:

$\{x > 0\}\ x := x + 3\ ;\ x := x + 1\ \{x > 4\}$

Assignment axiom (schema)

$$\{P\,[e\,/\,x]\} \quad x := e \quad \{P\}$$

$P\,[e/x]$ is the expression obtained from P by replacing (substituting) every occurrence of x by e.

Substitution

x [x/x] =

x [y/x] =

x [x/y] =

x [z/y] =

3 * x + 1 [y/x] =

Applying the assignment axiom

$\{y > z - 2\}\ x := x + 1\ \{y > z - 2\}$

$\{2 + 2 = 5\}\ x := x + 1\ \{2 + 2 = 5\}$

$\{y > 0\}\ x := y\ \{x > 0\}$

$\{x + 1 > 0\}\ x := x + 1\ \{x > 0\}$

Limits to the assignment axiom

No side effects in expressions!

```
asking_for_trouble (x: in out INTEGER): INTEGER
    do
        x := x + 1;
        global := global + 1;
        Result := 0
    end
```

Do the following hold?

```
{global = 0}  u := asking_for_trouble (a)   {global = 0}
{a = 0}       u := asking_for_trouble (a)   {a = 0}
```

The rule of constancy

$$\frac{\{P\}\ A\ \{Q\},\quad FV(R) \cap \text{modifies}(A) = \varnothing}{\{P \text{ and } R\}\ \ A\ \ \{Q \text{ and } R\}}$$

FV(F) = variables free in formula F

modifies(A) = variables assigned to in code A

“Whatever A doesn’t modify
stays the same”

The rule of constancy: examples

$\{ y = 3 \}\ x := x + 1\ \{ y = 3 \}$

$\{ \forall y \neq 0\!: y^2 > 0 \}\ y := y + 1\ \{ \forall y \neq 0\!: y^2 > 0 \}$

$\{ y = 3 \}\ x := sqrt(y)\ \{ y = 3 \}$

$\{ a[3] = 0 \}\ a[i] := 2\ \{ a[3] = 0 \}$

$\{ bob.age = 65 \}\ tony.age := 78\ \{ bob.age = 65 \}$

The rule of constancy: caveats

$\{ y = 3 \}\ x := x + 1\ \{ y = 3 \}$

$\{ \forall y \neq 0: y^2 > 0 \}\ y := y + 1\ \{ \forall y \neq 0: y^2 > 0 \}$

$\{ y = 3 \}\ x := sqrt(y)\ \{ y = 3 \}$

Only if sqrt doesn't have side effects on y!

$\{ a[3] = 0 \}\ a[i] := 2\ \{ a[3] = 0 \}$

Only if $i \neq 3$!

$\{ bob.age = 65 \}\ tony.age := 78\ \{ bob.age = 65 \}$

Only if $bob \neq tony$, i.e., they are not aliases!

The assignment axiom for arrays

{ P [if k = i then e else a[k] / a[k]] } a[i] := e { P }

Example:

{ 3 = i or (3 ≠ i and a[3] = 2) }

a[i] := 2

{ a[3] = 2 }

Conditional rule

$$\frac{\{P \ \textbf{and}\ c\}\ A\ \{Q\}, \qquad \{P\ \textbf{and not}\ c\}\ B\ \{Q\}}{\{P\}\ \ \textbf{if}\ c\ \textbf{then}\ A\ \textbf{else}\ B\ \textbf{end}\ \{Q\}}$$

Example:

$\{y > 0\}$

if $x > 0$ then $y := y + x$ else $y := y - x$

$\{y > 0\}$

Conditional rule: example proof

Prove:

$\{ m, n, x, y > 0$ **and** $x \neq y$ **and** $gcd(x, y) = gcd(m, n) \}$

if $x > y$ **then**

$\quad x := x - y$

else

$\quad y := y - x$

end

$\{ m, n, x, y > 0$ **and** $gcd(x, y) = gcd(m, n) \}$

Loop rule (partial correctness)

{P} A {I}   {I **and not** c} B {I}

{P}   **from** A **until** c **loop** B **end** {I **and** c}

{P} A {I} proves initiation: the invariant holds initially

{I **and not** c} B {I} proves consecution (or inductiveness): the invariant is preserved by an arbitrary iteration of the loop

Loop rule (partial correctness, variant)

$$\frac{\{P\}\ A\ \{I\},\quad \{I\ \textbf{and not}\ c\}\ B\ \{I\},\quad \{(I\ \textbf{and}\ c)\ \textbf{implies}\ Q\}}{\{P\}\quad \textbf{from}\ A\ \textbf{until}\ c\ \textbf{loop}\ B\ \ \textbf{end}\quad \{Q\}}$$

Example:

```
{y > 3 and n > 0}
        from i := 0 until i = n loop
                i := i + 1
                y := y + 1
        end
{y > 3 + n}
```

Loop termination

Must show there is a variant:

An expression v of type INTEGER such that
(for a loop **from** A **until** c **loop** B **end** with precondition P):

1. {P} A {v ≥ 0}
2. {v ≥ 0} is an invariant of the loop
3. v decreases with every iteration:
   ∀ v0 > 0: {v = v0 **and not** c} B {v < v0}

You can reuse invariants used for partial correctness
to prove 1, 2, and 3.

Loop termination: example

```
{y > 3 and n > 0}
        from i := 0 until i = n loop
                i := i + 1
                y := y + 1
        variant
                ??
        end
{y > 3 + n}
```

Computing the maximum of an array

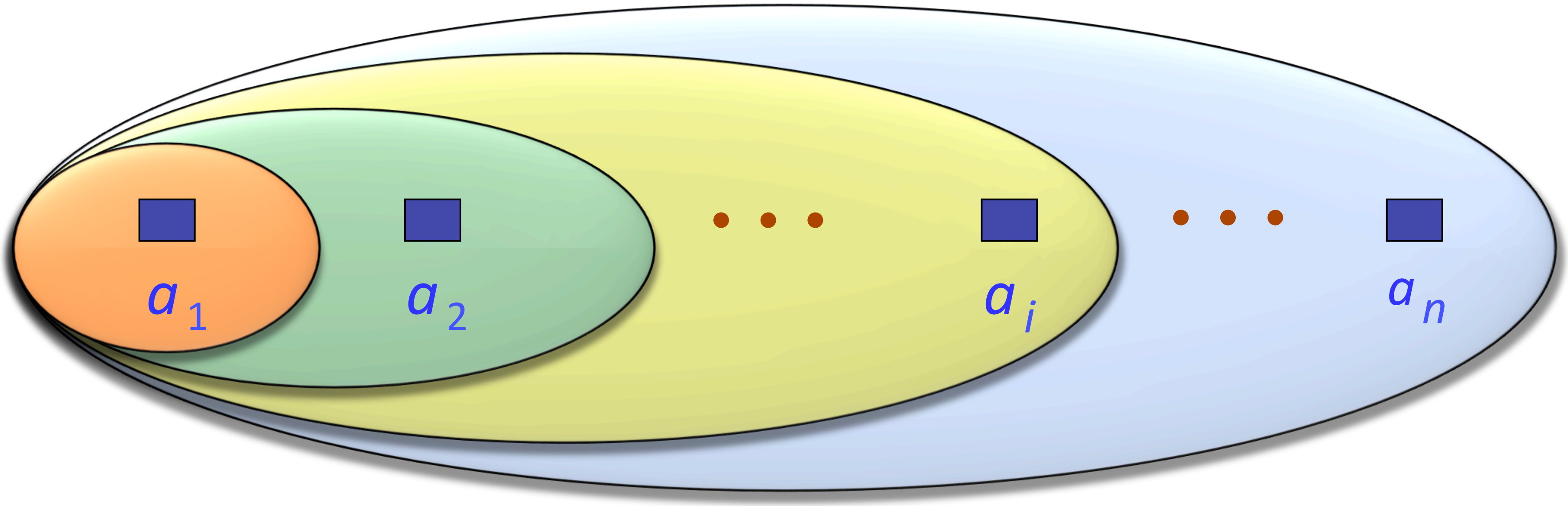
```
from
        i := 0 ; Result := a [1]
until

        i = a.upper
loop

        i := i + 1

        Result := max (Result , a [i ])
end
```

Loop as approximation strategy



Loop body:

$i := i + 1$
Result := *max* (**Result** , *a* [*i*])

Result = a_1 = Max (a_1 .. a_1)

Result = Max (a_1 .. a_2)

Result = Max (a_1 .. a_i)

The loop invariant

Result = Max (a_1 .. a_n)